http://people.sc.fsu.edu/~jburkardt/presentations/asa_2011_geometry_homework1.pdf

Homework #12

Algorithms for Science Applications II
Assigned: Friday, 15 April 2011
Due: Friday, 22 April 2011

Problem 1: *The Barycentric Coordinates of a Point on a Line*

A point **P** has *barycentric coordinates* (α, β) with respect to points **A** and **B** if

$$P(\alpha, \beta) = \alpha \cdot A + \beta \cdot B$$

where the scalars α and β have the property $\alpha + \beta = 1$

For convenience, we can replace β by the value $1 - \alpha$. The points **A**, **B** and **P** can be in a 1-dimensional space, or a plane, or higher-dimensional space. As long as **A** and **B** are distinct, the set of all possible points $P(\alpha) = \alpha \cdot A + (1 - \alpha) \cdot B$ describes a line.

Given α, it's easy to determine $P(\alpha)$. On the other hand, given a point **P**, we can go backwards and determine the corresponding α, as long, that is, as **P** actually does lie on the line through **A** and **B**. Think about how to do this, and then determine α for the following cases:

A	B	P(α)	Value of α
1	4	3	----------
(1,2)	(5,1)	(401,-98)	----------
(1,2,3)	(5,7,2)	(40,52,-7)	----------

Notice that **A** has coordinate 0, **B** coordinate 1, points in between have coordinates between 0 and 1. This means that the barycentric coordinate system gives a natural parameterization of lines in any dimension that is easy to understand and work with.

Turn in: Your table of α values.

Problem 2: *The Barycentric Coordinates of a Point in a Triangle*

Now suppose we have a triangle with vertices **A**, **B** and **C**. A point **P** has *barycentric coordinates* (α, β, γ) with respect to the triangle if

$$P(\alpha, \beta, \gamma) = \alpha \cdot A + \beta \cdot B + \gamma \cdot C$$

where we also require $\alpha + \beta + \gamma = 1$ Again, because the coordinates have to sum to 1, we can replace the γ coordinate by the value $1 - \alpha - \beta$, reducing our variables to 2.

As long as the vertices are distinct, the set of all possible points $P(\alpha, \beta)$ now describes a plane. What is more interesting, however, is that, is what happens if we require all coordinates to be positive:

$$0 \le \alpha$$
$$0 \le \beta$$
$$0 \le \gamma \iff \alpha + \beta \le 1$$

The set of points $P(\alpha, \beta)$ generated under these conditions is the triangle ABC.

Given the Cartesian coordinates of the vertices of a triangle and the barycentric coordinates of a point, it's easy to determine the Cartesian coordinates of a point from the formula. Now we ask the inverse question: given a point P, can we determine its barycentric coordinates (α, β)?

In order to understand how to solve this problem, let's write out the equations that generate the point from its (unknown) coordinates. Here, we will assume that the vertices lie in the two-dimensional plane.

$$\begin{pmatrix} A(x) & B(x) & C(x) \\ A(y) & B(y) & C(y) \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ 1-\alpha-\beta \end{pmatrix} = \begin{pmatrix} P(x) \\ P(y) \end{pmatrix}$$

can be rewritten as the 2x2 linear system:

$$\begin{pmatrix} A(x)-C(x) & B(x)-C(x) \\ A(y)-C(y) & B(y)-C(y) \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} P(x) \\ P(y) \end{pmatrix}$$

and, under the assumption that the vertices are distinct, this linear system can be solved for the coordinates.

Suppose we have the triangle ABC with vertices A=(4,0), B=(3,4), C=(0,1). Use the suggested technique to determine the barycentric coordinates of the following points. Based on the values of the coordinates, which points are inside the triangle?

P	α	β	γ	Inside triangle?
(2,1)	----------	----------	----------	----------
(4,1)	----------	----------	----------	----------
(3,2)	----------	----------	----------	----------
(2,3)	----------	----------	----------	----------
(1,4)	----------	----------	----------	----------
(3,4)	----------	----------	----------	----------
(3,5)	----------	----------	----------	----------
(4,5)	----------	----------	----------	----------

Turn in: Your completed table of coordinate values.