Lectures - Week 14
Vector Form of Taylor's Series, Integration in Higher Dimensions, and Green's Theorems

Vector form of Taylor Series

We have seen how to write Taylor series for a function of two independent variables, i.e., to expand $f(x, y)$ in the neighborhood of a point, say (a, b). We can write out the terms through the second derivative explicitly, but it's difficult to write a general form. Recall that we have

$$
\begin{aligned}
f(a+\Delta x, b+\Delta y) = {} & f(a,b) + \Delta x f_x(a,b) + \Delta y f_y(a,b) \\
& + \frac{\Delta x^2}{2} f_{xx}(a,b) + \frac{\Delta y^2}{2} f_{yy}(a,b) + \Delta x \Delta y f_{xy}(a,b) + \cdots
\end{aligned}
$$

or equivalently as

$$
\begin{aligned}
f(x,y) = {} & f(a,b) + (x-a) f_x(a,b) + (y-b) f_y(a,b) \\
& + \frac{(x-a)^2}{2} f_{xx}(a,b) + \frac{(y-b)^2}{2} f_{yy}(a,b) + (x-a)(y-b) f_{xy}(a,b) + \cdots
\end{aligned}
$$

where $x = a + \Delta x$ so $\Delta x = x - a$ and $y = b + \Delta y$.

We now want to see if there is a simple way to write these terms for expanding $f(\vec{x})$ where $\vec{x} = (x_1, x_2, \ldots, x_n)^T$ in terms of the operator ∇. Before we do this, lets first rewrite the above Taylor series expansion for $f(x, y)$ in vector form and then it should be straightforward to see the result if f is a function of more than two variables. We let $\vec{x} = (x, y)$ and $\vec{a} = (a, b)$ be the point we are expanding $f(\vec{x})$ about. Now the term representing the change becomes the vector $\vec{x} - \vec{a} = (x - a, y - b)^T$. The gradient of f, ∇f, is just $(f_x, f_y)^T$ so the terms involving the first derivatives are just the dot product of $(\vec{x} - \vec{a})$ and ∇f. We claim that the terms involving the second derivatives are found by taking the product

$$
\frac{1}{2}\Big[(\vec{x}-\vec{a})^T \begin{pmatrix} f_{xx}(\vec{a}) & f_{xy}(\vec{a}) \\ f_{yx}(\vec{a}) & f_{yy}(\vec{a}) \end{pmatrix} (\vec{x}-\vec{a})\Big]
$$

because

$$
\begin{pmatrix} f_{xx}(\vec{a}) & f_{xy}(\vec{a}) \\ f_{yx}(\vec{a}) & f_{yy}(\vec{a}) \end{pmatrix} \begin{pmatrix} x-a \\ y-b \end{pmatrix} = \begin{pmatrix} (x-a) f_{xx}(a,b) + (y-b) f_{xy}(a,b) \\ (x-a) f_{yx}(a,b) + (y-b) f_{yy}(a,b) \end{pmatrix}
$$

and

$$
\frac{1}{2} \begin{pmatrix} x-a & y-b \end{pmatrix} \begin{pmatrix} (x-a) f_{xx}(a,b) + (y-b) f_{xy}(a,b) \\ (x-a) f_{yx}(a,b) + (y-b) f_{yy}(a,b) \end{pmatrix}
$$

$$
= \frac{1}{2}\Big[(x-a)^2 f_{xx}(a,b) + (x-a)(y-b) f_{xy}(a,b) + (y-b)(x-a) f_{yx}(a,b) + (y-b)^2 f_{yy}(a,b)\Big]
$$

which is just our terms in the Taylor series above assuming that $f_{xy} = f_{yx}$. If we note that the matrix above is just the Hessian of $f(x, y)$ then we can generalize our result for a function of more than two independent variables. We can write the Taylor series expansion for a function $f(\vec{x}$ where $\vec{x} \in \mathrm{R}^n$ in the neighborhood of the point $\vec{a}$ as

$$f(\vec{x}) = f(\vec{a}) + (\vec{x} - \vec{a})^T \nabla f(\vec{a}) + \frac{1}{2!}(\vec{x} - \vec{a})^T H_f(\vec{a})(\vec{x} - \vec{a}) + \cdots$$

where $H_f(\vec{a})$ denotes the Hessian of f evaluated at $\vec{a}$. Higher order terms can be written in terms of tensors but we will not go in to that here.

Multiple Integration

Recall from calculus of a single variable that if $f(x) \geq 0$ on $[a, b]$ then the integral $\int_a^b f(x)\, dx$ graphically represents the area under the curve $y = f(x)$, above the x axis and between the lines $x = a$ and $x = b$. An analogous result holds for the volume of a region in R^3 below the curve $z = f(x, y)$ and between the lines $x = a$, $x = b$, $y = c$ and $y = d$. In this case we have a double integral. If R represents the rectangle $a \leq x \leq b$, $c \leq y \leq d$ then we can write

$$\int_R f(x, y)\, dA \qquad \text{or} \qquad \int_c^d \int_a^b f(x, y)\, dxdy$$

How do we evaluate such integrals? We can write this integral as an *iterated integral* of the form

$$\int_c^d \Big[\int_a^b f(x, y)\, dx \Big] dy$$

that is, we evaluate the inner integral treating y as a constant and then the outer integral.

Example Evaluate $\int_0^2 \int_0^3 (2 - y)\, dx\, dy$. We have

$$\int_0^2 \Big[\int_0^3 (2 - y)\, dx \Big]\, dy = \int_0^2 \Big[(2x - xy) \Big|_0^3 \Big]\, dy = \int_0^2 \Big[6 - 3y - 0 \Big]\, dy = 6y - \frac{3}{2}y^2 \Big|_0^2 = 6$$

Of course our domain is not always a rectangle and so we may need to map it into a simpler region and evaluate the integral there. For example, if we wanted to evaluate an integral over a domain where polar coordinates make the integrand much simpler, we might want to change to polar coordinates and map our domain there. We have done this for a function of a single variable. For example, to integrate $\int_0^2 xe^{x^2}\, dx$ we use the substitution $u = x^2$ and $du = 2xdx$ to write

$$\int_0^2 xe^{x^2}\, dx = \int_0^4 \frac{1}{2} e^u\, dx = \frac{1}{2} e^u \Big|_0^4 = \frac{1}{2}\Big(e^4 - 1\Big)$$

What we are really doing here is using the relationship

$$(*) \qquad \int_a^b f(x)\, dx = \int_c^d f\Big(g(u)\Big) g'(u)\, du$$

Here we have the generic substitution $x = g(u)$ and $dx = g'(u)du$. In our example $u = x^2$ so $x = \sqrt{u}$ and thus $g(u) = \sqrt{u}$. Also $dx = g'(u)du$ is just $dx = \frac{1}{2}(u)^{-1/2}du = du/2x$ or $du = 2xdx$.

We would like an analogue of (*) in higher dimensions. Note that if we are in two dimensions, then we need to transform both x and y variables by say

$$x = g(u,v) \qquad y = h(u,v)$$

We assume that we have in hand a change of variables and now want to write the integral $\int\int_D f(x,y)\, dxdy$ in terms of u, v. We have the following relationship analogous to (*).

Theorem Let $f(x,y) \in C(\mathrm{R}^2)$ and let T be a one to one mapping $T : D \to D^*$ in the $u, v-$ plane using $x = g(u,v)$, $y = h(u,v)$. Then

$$(**) \qquad \int\int_D f(x,y)\, dxdy = \int\int_{D^*} f\big[g(u,v), h(u,v)\big]\big|\det J(u,v)\big|\, du\, dv$$

Here $J(u,v)$ is the Jacobian

$$J(u,v) = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

and we take the magnitude of its determinant in (**).

Example Let D be the ellipse $\frac{x^2}{z^2} + \frac{y^2}{b^2} = 1$. Calculate the area of the ellipse by mapping it to the unit circle D^*, $u^2 + v^2 = 1$.

We want to calculate the area by evaluating $\int\int_D dxdy$ by using the mapping $u = x/a$, $v = y/b$. So in our relationship in (**) we have $x = g(u,v) = au$ and $y = h(u,v) = bv$. The Jacobian is given by

$$J(u,v) = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \Rightarrow \det J = ab$$

Our integral becomes

$$\int\int_{D^*} |ab|\, dudv = |ab| \int\int_{D^*} dudv = (\text{area of } D^*)|ab| = \pi|ab|$$

which is the area of our ellipse. Here we don't actually need the absolute values around ab because them are positive because they represent the length of the major and minor axes.

Example Transform the integral $\int\int_D f(x,y)\, dxdy$ into polar coordinates (r,θ) by using the mapping

$$x = r\cos\theta \qquad y = r\sin\theta\,.$$

Here $x = g(r,\theta) = r\cos\theta$ and $y = h(r,\theta) = r\sin\theta$. We compute the Jacobian and its determinant as

$$J(r,\theta) = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \Rightarrow \det J = r\cos^2\theta + r\sin^2\theta = r(\cos^2\theta + \sin^2\theta) = r$$

Thus the change of variables gives

$$\int\int_D f(x,y)\,dxdy = \int\int_{D^*} f(r\cos\theta, r\sin\theta) r dr d\theta$$

Example Write the integral

$$\int\int_D e^{-x^2/4-y^2/9}\,dydx$$

in terms of polar coordinates.
Here we need to include a constant so the transformed integrand is e^{-r^2}; we take $x = 2r\cos\theta$ and $y = 3r\sin\theta$ and thus the Jacobian is

$$J = \begin{pmatrix} 2\cos\theta & -2r\sin\theta \\ 3\sin\theta & 3r\cos\theta \end{pmatrix} \Rightarrow \det J = 6r$$

so we have

$$\int\int_D e^{-x^2/4-y^2/9}\,dydx = 6\int\int_{D^*} e^{-r^2} r\,drd\theta$$

where D^* is found through our mapping.

Multiple integrals in higher dimensions have an analogous definition.