Mass conservation for NSE

Reference: Michael Case, Vincent Ervin, Alexander Linke, Leo Rebholz, "A connection between Scott-Vogelius and grad-div stabilized Taylor-Hood FE approximations of the Navier-Stokes equations," *SIAM J. Numer. Anal.*, **49**, no. 4 (2011), pp. 1461-1481.

The authors present a study of mass conservation in a finite element simulation of fluid-flow (incompressible Navier-Stokes) using two different classes of element. One of the classes of element they used is Taylor-Hood, amounting to the usual P^2 element for velocity and P^1 for pressure. The other class of element, Scott-Vogelius, is not available inside either FreeFem++ or FEniCS. For this project, you will be repeating one of the numerical examples discussed in Section 5.1 (really 5.2) in the paper.

The time-dependent, incompressible Navier-Stokes equations are given as

$$\frac{\partial u}{\partial t} - \nu\Delta u + u \cdot \nabla u + \nabla p = f \text{ in } \Omega \times (0,T] \qquad \text{(NSE)}$$

$$\nabla \cdot u = 0 \text{ in } \Omega \times (0,T] \qquad \text{(Mass Cons)}$$

$$u(x,0) = u_0(x) \text{ in } \Omega$$

$$u = 0 \text{ on } \partial\Omega \times (0,T]$$

These equations appear as (2.1)-(2.4) in the paper. The equation (Mass Cons) is a form of conservation of mass for incompressible fluids. The point of departure for this paper is that, while the equation (Mass Cons) is satisfied in the mean, it can be far from satisfied on a per-element basis.

The weak form of this system, including a "grad-div stabilization term" $(\gamma(\nabla \cdot u_h, \nabla \cdot v_h))$, is given as equations (2.8) and (2.9) in the paper. In Section 5.1, a numerical experiment based on 2D flow around a cylinder is presented. This experiment examines the effect of γ on the solution for both Taylor-Hood and Scott-Vogelius elements. In each case, the norm of the divergence, $\|\nabla \cdot u_h^n\|$ is computed. A small value represents "good" conservation of mass and a large norm represents "bad" conservation of mass.

In this project, you will work *only* with Taylor-Hood elements and you will generate a mesh without being concerned whether it is barycentric or not. You will be interested in reproducing only the Taylor-Hood portions of the experiment. (Section 5.2 discusses this case, and, in fact, you are reproducing that experiment and not the one in Section 5.1, but the description in Section 5.1 is complete.)

You are to reproduce this experiment as best you can. The experiment involves 6578 velocity degrees of freedom and 845 pressure degrees of freedom (two velocity dofs per node). You are to generate a mesh with somewhere around 6500 velocity degrees of freedom and however many pressure degrees of freedom go with it. If it turns out that this is too many for you to handle on your computer, you are free to drop back to a smaller number, but it is not likely that you will get good results with fewer than 5000 velocity degrees of freedom. You should compute only to a final time of $t = 7$ to save yourself some computing time. Present your results in figures roughly similar to Figures 2 and 3 in the paper.