FreeFem++, part II

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Topics

Syntax

Tutorial examples from Chapter 3
- Section 3.3 Acoustics
- Exercise 25
- Section 3.4 Thermal convection
  - Section 3.4.2 Radiation boundary condition
- Section 3.5 Heated fan blade
- Section 3.6: Rotating hill
- Syntax of macros
- Section 3.7: Elasticity
- Section 3.8: Stokes flow
- Section 3.9: Chorin's projection for NSE

Array operations: questions

```
real[int]  a(5),b(5),c(5),d(5);
a = 1;
b = 2;
c = 3;
a[2]=0;

d = ( a ? b : c );
cout << "d=" << d << endl;
cout << "==    2       2       3       2       2 " << endl;

d = (a ? 1 : 10);
cout << " (a ? 1 : 10) "<< d << endl;
d = (a ? b : -1);
cout << " (a ? b : -1 ) "<< d << endl;
d = (a ? -2 : c);
cout << " (a ? -2 : c) " <<  d << endl;
```

Output:

```
d=5
         2       2       3       2       2

==    2       2       3       2       2
 (a ? 1 : 10) 5
         1       1      10       1       1

 (a ? b : -1 ) 5
         2       2      -1       2       2

 (a ? -2 : c) 5
        -2      -2       3      -2      -2
```

Matlab-like colon notation

- The book says colon notation works for subscripts “like Matlab”
- The following code:

```
int N=5;
real[int] a(N),b(N),c(N);
a =1;
a(0:5:2) = 2;
cout <<"a = " << a << endl;
```

- Produces

```
a = 5
          2        1        2        1        2
```

- `a(0:4:2) = 2;` produces `2 1 2 1 1`
- `a(0:4:1) = 2;` produces `2 2 2 2 2`
- *Stay away from this syntax!*

1D array operations

```
int N=5;
real[int] a(N),b(N),c(N);
a = [1, -1, 2, 2, 5];    cout <<" a = " << a << endl;

b = a + a;      cout <<" b = a + a : " << b << endl;
b += a;         cout <<" b += a : " << b << endl;
b += 2*a;       cout <<" b += 2*a : " << b << endl;
b /= 2;         cout <<" b /= 2 : " << b << endl;
b .*= a;        cout << " b .*= a; b =" << b << endl;   // same b = b .* a
b ./= a;        cout << " b ./= a; b =" << b << endl;   // same b = b ./ a
c = a + b;      cout << " c = a + b : c=" << c << endl;
c = 2*a + 4*b; cout << " c =2*a + 4*b : c= " << c << endl;

c = a + 4*b;    cout << " c = a + 4b : c= " << c << endl;
c = -a + 4*b;   cout << " c = -a + 4b : c= " << c << endl;
c = -a - 4*b;   cout << " c = -a - 4b : c= " << c << endl;
c = -a - b;     cout << " c =-a - b : c= " << c << endl;

c = a .* b;     cout << " c =a .* b  : c= " << c << endl;
c = a ./ b;     cout << " c = a . /b  : c= " << c << endl;
c = 2 * b;      cout << " c = 2 * b   : c= " << c << endl;
c = b * 2 ;     cout << " c = b * 2   : c= " << c << endl;
```

Compressed output from previous code

```
a :                   5     1       -1       2       2       5
b = a + a :           5     2       -2       4       4      10
b += a :              5     3       -3       6       6      15
b += 2*a :            5     5       -5      10      10      25
b /= 2 :              5   2.5     -2.5       5       5    12.5
b .*= a; b =          5   2.5      2.5      10      10    62.5
b ./= a; b =          5   2.5     -2.5       5       5    12.5
c = a + b : c=        5   3.5     -3.5       7       7    17.5
c =2*a + 4*b : c=     5    12      -12      24      24      60
c = a + 4b : c=       5    11      -11      22      22      55
c = -a + 4b : c=      5     9       -9      18      18      45
c = -a - 4b : c=      5   -11       11     -22     -22     -55
c = -a - b : c=       5  -3.5     3.5      -7      -7   -17.5
c = a .* b  : c=      5   2.5      2.5      10      10    62.5
c = a . /b  : c=      5   0.4      0.4     0.4     0.4     0.4
c = 2 * b   : c=      5     5       -5      10      10      25
c = b * 2   : c=      5     5       -5      10      10      25
```

Array methods

```
int N=5;
real[int] a(N);
a = [1, -1, 2, 2, 5];
cout << " a = " << a << endl;

cout << " ||a||_1      = " <<  a.l1     << endl;
cout << " ||a||_2      = " <<  a.l2     << endl;
cout << " ||a||_infty = " <<  a.linfty << endl;
cout << " sum a_i      = " <<  a.sum    << endl;
cout << " max a_i      = " <<  a.max   << "  a[  " << a.imax << " ] = " << a[a.ima
cout << " min a_i      = " <<  a.min   << " a[  " << a.imin << " ] = " << a[a.imin
cout << " a'*a         = " <<  (a'*a)   << endl;
cout << " a.quantile(0.2) =  " <<  a.quantile(0.2) << endl;
cout << " a.sort =  " <<  a.sort << endl;  // changes a !
```

Results in

```
 a = 5
          1      -1       2       2       5

 ||a||_1      = 11
 ||a||_2      = 5.91608
 ||a||_infty = 5
 sum a_i      = 9
 max a_i      = 5  a[  4 ] = 5
 min a_i      = -1 a[  1 ] = -1
 a'*a         = 35
 a.quantile(0.2) =  1
 a = 5
         -1       1       2       2       5
```

Array mapping

```
int N=5;
real[int] a(N), b(N), c(N);
int[int] I=[2,3,4,-1,3];
a = [1, -1, 2, 2, 5];
cout << " a = " << a << endl;
cout << " I = " << I << endl;

b = c = -3;
b = a( I );    // for( i=0; i<b.n; i++) if( I[i] >=0 )  b[ i ]=a[ I[i] ];
c( I ) = a;    // for( i=0; i<I.n; i++) if( I[i] >=0 )  c( I[i] )=a[ i ];
cout << " b = a(I) : " << b << "\n  c(I) = a " << c << endl;
c( I ) += a;   // for( i=0;i<I.n;i++) if(I[i] >=0)  C(I[i])+=a[i];
cout << " b = a(I) : " << b << "\n  c(I) = a " << c << endl;
...............  RESULTS  ...................................
a = 5
          1       -1        2        2        5
I = 5
          2        3        4       -1        3
 b = a(I) : 5
          2        2        5       -3        2
  c(I) = a 5
         -3       -3        1        5        2
 b = a(I) : 5
          2        2        5       -3        2
  c(I) = a 5
         -3       -3        2        9        4
 b = a(I) : 5
          2        2        5       -3        2
  c(I) = a 5
         16       16       17       20       18
```

2D Arrays

```
int N=3,M=4;

real[int,int] A(N,M);
real[int]  b(N), c(M);
b = [1,2,3];
c = [4,5,6,7];

complex[int,int]  C(N,M);
complex[int]  cb=[1,2,3], cc=[10i,20i,30i,40i];

int [int] I = [2,0,1];
int [int] J = [2,0,1,3];

A = 1; // set the full matrix
cout << " I. A = " << A << endl;

A(2, :) = 4; //  the full row 2
cout << " II. A = " << A << endl;

A(:, 1) = 5; //  the full column 1
cout << " III. A = " << A << endl;

A(0:N-1, 2) = 2; // set the full column 2
cout << " IV. A = " << A << endl;

A(1, 0:2) = 3; // set row 1 from 0 to 2
cout << " A = " << A << endl;
```

Output from previous code

```
I. A = 3 4
           1   1   1   1
           1   1   1   1
           1   1   1   1

 II. A = 3 4
           1   1   1   1
           1   1   1   1
           4   4   4   4

 III. A = 3 4
           1   5   1   1
           1   5   1   1
           4   5   4   4

 IV. A = 3 4
           1   5   2   1
           1   5   2   1
           4   5   2   4

 A = 3 4
           1   5   2   1
           3   3   3   1
           4   5   2   4
```

Outer products

```
int N=3,M=4;

complex[int,int]  C(N,M);
complex[int]  cb=[1,2,3], cc=[10i,20i,30i,40i];

C   =   cb*cc';
C  +=   3*cb*cc';
C  -=   5i*cb*cc';
cout << " C = " << C << endl;
```

Results

```
C = 3 4
         (-50,-40) (-100,-80) (-150,-120) (-200,-160)
         (-100,-80) (-200,-160) (-300,-240) (-400,-320)
         (-150,-120) (-300,-240) (-450,-360) (-600,-480)
```

Sparse matrices

```
int N=3,M=4;

real[int,int] A(N,M);
real[int]  b(N), c(M);
b = [1,2,3];
c = [4,5,6,7];

int [int] I = [2,0,1];
int [int] J = [2,0,1,3];

A = 2.*b*c'; // outer product
cout << " A = " << A << endl;

// the way to transform a array to a sparse matrix
matrix B;

B = A;
cout << "B = A =  " << B << endl;

B = A(I,J);                         // B(i,j)= A(I(i),J(j))
B = A( I^-1 , J^-1 );               // B(I(i),J(j))= A(i,j)

B = b*c';                           // outer product  B(i,j)  = b(i)*c(j)
B = b*c';                           // outer product  B(i,j)  = b(i)*c(j)
B = (2*b*c')(I,J);                  // outer product  B(i,j)  = b(I(i))*c(J(j))
B = (3.*b*c')( I^-1 , J^-1 );       // outer product  B(I(i),J(j))   = b(i)*c(j)
```

Results

```
 A = 3 4
          8   10   12   14
         16   20   24   28
         24   30   36   42

B = A =  # Sparse Matrix (Morse)
# first line: n m (is symmetic) nbcoef
# after for each nonzero coefficient: i j a_ij where (i,j) \in 1,...,nx1,...,m
3 4 0  12
        1         1 8
        1         2 10
        1         3 12
        1         4 14
        2         1 16
        2         2 20
        2         3 24
        2         4 28
        3         1 24
        3         2 30
        3         3 36
        3         4 42
```

Elementary functions

```
int N=3,M=4;
real[int]  b(N), c(M);
b = [1,2,3];
c = [4,5,6,7];

complex[int]  cb=[1,2,3], cc=[10i,20i,30i,40i];

cout << " b =" <<  b << endl;
b = exp(b) ;
cout << " exp(b) =" <<  b << endl;
cb += complex(10.)*cc(0:2);
cout << " cb =" <<  cb << endl;
cb = exp(cb) ;
cout << " exp(cb) =" <<  cb << endl;
cout << " exp(cb).re =" <<  cb.re << endl;
cout << " exp(cb).im =" <<  cb.im << endl;
```

Results

```
b =3
          1        2        3
exp(b) =3
        2.718281828     7.389056099     20.08553692
cb =3
        (1,100) (2,200) (3,300)
exp(cb) =3
   (2.3440257,-1.3764445)  (3.5998571,-6.452843)  (-0.44382246,-20.080633)
exp(cb).re =3
        2.344025721     3.599857061     -0.4438224624
exp(cb).im =3
        -1.376444521    -6.45284272     -20.08063284
```

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Example 23, sound propagation

- Sound in air satisfies

$$\frac{\partial^2 u}{\partial t^2} - c^2 \Delta u = 0$$

- If monochromatic boundary conditions $g(x)e^{ikt}$
- Then solution $u(x,t) = v(x)e^{ikt}$
- v is a solution of Helmholz's equation

$$k^2 v + c^2 \Delta v = 0 \text{ in } \Omega$$
$$\left.\frac{\partial v}{\partial n}\right|_{\Gamma} = g$$

- Eigenvalues of Laplacian are negative!
- Real eigenvalues of Helmholz $\implies$ possible singularity!

example23.edp sound pressure code

```
verbosity=10;  // increase informational printing
real konc2;   // (k/c)^2
if (true)
  konc2 = 1.0;  // good solution
 else
  konc2 = 14.6993;  // eigenvalue! => bad solution


func g = y*(1-y);  // sound source
```

example23.edp sound pressure code

```
verbosity=10;  // increase informational printing
real konc2;   //  (k/c)^2
if (true)
  konc2 = 1.0;  // good solution
 else
  konc2 = 14.6993;  // eigenvalue! => bad solution


func g = y*(1-y);  // sound source

border a0(t=0,1) { x= 5; y= 1+2*t; }
border a1(t=0,1) { x=5-2*t; y= 3; }
border a2(t=0,1) { x= 3-2*t; y=3-2*t; }
border a3(t=0,1) { x= 1-t; y= 1; }
border a4(t=0,1) { x= 0; y= 1-t; }
border a5(t=0,1) { x= t; y= 0; }
border a6(t=0,1) { x= 1+4*t; y= t; }

mesh Th=buildmesh( a0(20) + a1(20) + a2(20)
        + a3(20) + a4(20) + a5(20) + a6(20));
```

example23.edp sound pressure code

```
verbosity=10;  // increase informational printing
real konc2;    // (k/c)^2
if (true)
  konc2 = 1.0;   // good solution
 else
  konc2 = 14.6993;  // eigenvalue! => bad solution


func g = y*(1-y);  // sound source

border a0(t=0,1) { x= 5; y= 1+2*t; }
border a1(t=0,1) { x=5-2*t; y= 3; }
border a2(t=0,1) { x= 3-2*t; y=3-2*t; }
border a3(t=0,1) { x= 1-t; y= 1; }
border a4(t=0,1) { x= 0; y= 1-t; }
border a5(t=0,1) { x= t; y= 0; }
border a6(t=0,1) { x= 1+4*t; y= t; }

mesh Th=buildmesh( a0(20) + a1(20) + a2(20)
        + a3(20) + a4(20) + a5(20) + a6(20));

fespace Vh(Th,P1);
Vh u,v;
```

example23.edp sound pressure code

```
verbosity=10;  // increase informational printing
real konc2;    // (k/c)^2
if (true)
  konc2 = 1.0;   // good solution
 else
  konc2 = 14.6993;  // eigenvalue! => bad solution


func g = y*(1-y);  // sound source

border a0(t=0,1) { x= 5; y= 1+2*t; }
border a1(t=0,1) { x=5-2*t; y= 3; }
border a2(t=0,1) { x= 3-2*t; y=3-2*t; }
border a3(t=0,1) { x= 1-t; y= 1; }
border a4(t=0,1) { x= 0; y= 1-t; }
border a5(t=0,1) { x= t; y= 0; }
border a6(t=0,1) { x= 1+4*t; y= t; }

mesh Th=buildmesh( a0(20) + a1(20) + a2(20)
        + a3(20) + a4(20) + a5(20) + a6(20));

fespace Vh(Th,P1);
Vh u,v;

solve sound(u,v)=int2d(Th)(u*v * konc2 - dx(u)*dx(v) - dy(u)*dy(v))
                 - int1d(Th,a4)(g*v);
```

example23.edp sound pressure code

```
verbosity=10;  // increase informational printing
real konc2;    // (k/c)^2
if (true)
  konc2 = 1.0;   // good solution
 else
  konc2 = 14.6993;  // eigenvalue! => bad solution


func g = y*(1-y);  // sound source

border a0(t=0,1) { x= 5; y= 1+2*t; }
border a1(t=0,1) { x=5-2*t; y= 3; }
border a2(t=0,1) { x= 3-2*t; y=3-2*t; }
border a3(t=0,1) { x= 1-t; y= 1; }
border a4(t=0,1) { x= 0; y= 1-t; }
border a5(t=0,1) { x= t; y= 0; }
border a6(t=0,1) { x= 1+4*t; y= t; }

mesh Th=buildmesh( a0(20) + a1(20) + a2(20)
        + a3(20) + a4(20) + a5(20) + a6(20));

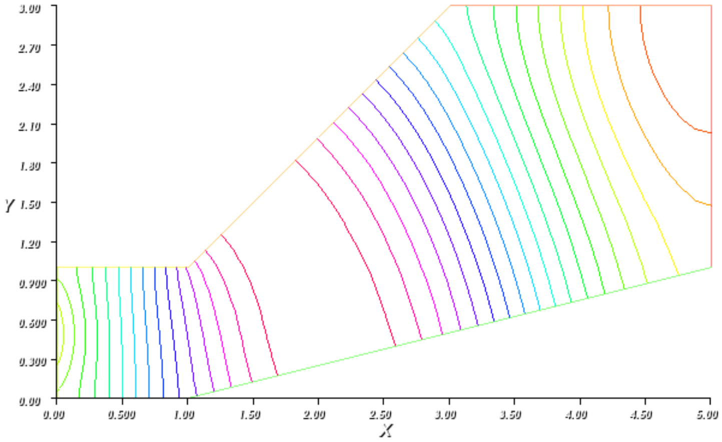
fespace Vh(Th,P1);
Vh u,v;

solve sound(u,v)=int2d(Th)(u*v * konc2 - dx(u)*dx(v) - dy(u)*dy(v))
                 - int1d(Th,a4)(g*v);

plot(u, wait=1);
```

`example23.edp` Sound pressure results

Maximum pressure = 0.18



example23.edp Eigenpairs code

```
Vh u1,u2;
```

`example23.edp` Eigenpairs code

```
Vh u1,u2;

real sigma = 1;  // value of the shift
```

example23.edp Eigenpairs code

```
Vh u1,u2;

real sigma = 1;  // value of the shift

// OP = A - sigma B ;  //  the shifted matrix

varf  op(u1,u2)= int2d(Th)(  dx(u1)*dx(u2) + dy(u1)*dy(u2) - sigma* u1*u2 );
varf b([u1],[u2]) = int2d(Th)(  u1*u2 );
```

example23.edp Eigenpairs code

```
Vh u1,u2;

real sigma = 1;  // value of the shift

// OP = A - sigma B ;  //  the shifted matrix

varf  op(u1,u2)= int2d(Th)(  dx(u1)*dx(u2) + dy(u1)*dy(u2) - sigma* u1*u2 );
varf b([u1],[u2]) = int2d(Th)(  u1*u2 );

matrix OP = op(Vh, Vh, solver=UMFPACK);
matrix B = b(Vh, Vh, solver=UMFPACK);
```

example23.edp Eigenpairs code

```
Vh u1,u2;

real sigma = 1;  // value of the shift

// OP = A - sigma B ;  //  the shifted matrix

varf  op(u1,u2)= int2d(Th)(  dx(u1)*dx(u2) + dy(u1)*dy(u2) - sigma* u1*u2 );
varf b([u1],[u2]) = int2d(Th)(  u1*u2 );

matrix OP = op(Vh, Vh, solver=UMFPACK);
matrix B = b(Vh, Vh, solver=UMFPACK);

int nev=20;  // number of computed eigen value close to sigma

real[int] ev(nev); // to store the  nev eigenvalue
Vh[int] eV(nev);   // to store the nev eigenVector
```

`example23.edp` Eigenpairs code

```
Vh u1,u2;

real sigma = 1;  // value of the shift

// OP = A - sigma B ;  //  the shifted matrix

varf  op(u1,u2)= int2d(Th)(  dx(u1)*dx(u2) + dy(u1)*dy(u2) - sigma* u1*u2 );
varf b([u1],[u2]) = int2d(Th)(  u1*u2 );

matrix OP = op(Vh, Vh, solver=UMFPACK);
matrix B = b(Vh, Vh, solver=UMFPACK);

int nev=20;  // number of computed eigen value close to sigma

real[int] ev(nev); // to store the  nev eigenvalue
Vh[int] eV(nev);   // to store the nev eigenVector

int k=EigenValue(OP, B, sym=true, sigma=sigma, value=ev, vector=eV,
                   tol=1e-10, maxit=0, ncv=0);

assert(k == nev);  // k is number of eigenvalues found
```

example23.edp Eigenpairs code

```
Vh u1,u2;

real sigma = 1;  // value of the shift

// OP = A - sigma B ;  //  the shifted matrix

varf  op(u1,u2)= int2d(Th)(  dx(u1)*dx(u2) + dy(u1)*dy(u2) - sigma* u1*u2 );
varf b([u1],[u2]) = int2d(Th)(  u1*u2 );

matrix OP = op(Vh, Vh, solver=UMFPACK);
matrix B = b(Vh, Vh, solver=UMFPACK);

int nev=20;  // number of computed eigen value close to sigma

real[int] ev(nev); // to store the  nev eigenvalue
Vh[int] eV(nev);   // to store the nev eigenVector

int k=EigenValue(OP, B, sym=true, sigma=sigma, value=ev, vector=eV,
                   tol=1e-10, maxit=0, ncv=0);

assert(k == nev);  // k is number of eigenvalues found

for (int n=0; n<nev; n++){
  cout << "Eigenvalue " << n << " = " << ev[n] << endl;
}

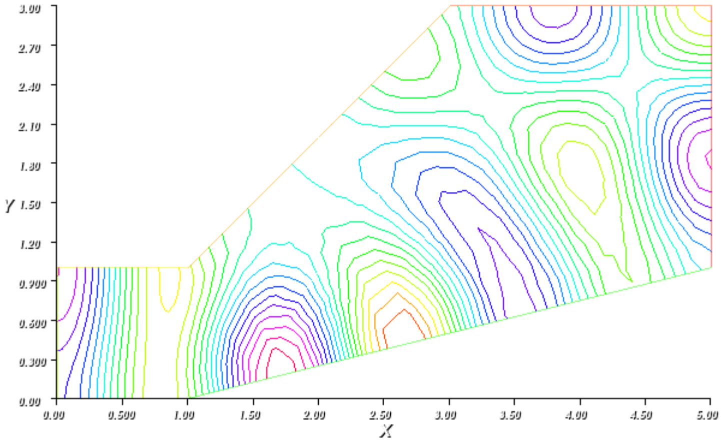
for (int n=0; n<nev; n++){
  plot(eV[n],wait=1);
}
```

example23.edp eigenvalues

```
Real symmetric eigenvalue problem: A*x - B*x*lambda
Eigenvalue 0 = -2.22045e-15
Eigenvalue 1 = 0.395913
Eigenvalue 2 = 1.39024
Eigenvalue 3 = 2.2087
Eigenvalue 4 = 2.97306
Eigenvalue 5 = 4.37777
Eigenvalue 6 = 5.13117
Eigenvalue 7 = 7.05942
Eigenvalue 8 = 7.94178
Eigenvalue 9 = 8.61535
Eigenvalue 10 = 10.2941
Eigenvalue 11 = 10.7342
Eigenvalue 12 = 11.421
Eigenvalue 13 = 12.4144
Eigenvalue 14 = 14.6993
Eigenvalue 15 = 14.9511
Eigenvalue 16 = 16.7588
Eigenvalue 17 = 18.5701
Eigenvalue 18 = 19.4472
Eigenvalue 19 = 20.0441
times: compile 0.02s, execution 1.14s,  mpirank:0
```

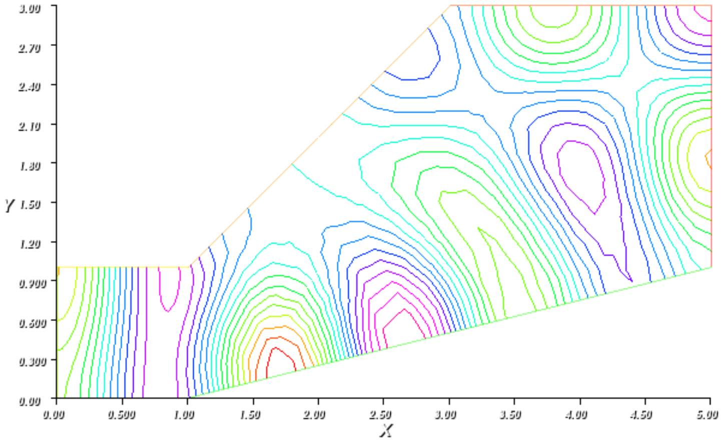
`example23.edp` fourteenth eigenvector

Fourteenth eigenvalue = 14.6993



`example23.edp` Solution with `konc2 = 14.6993`

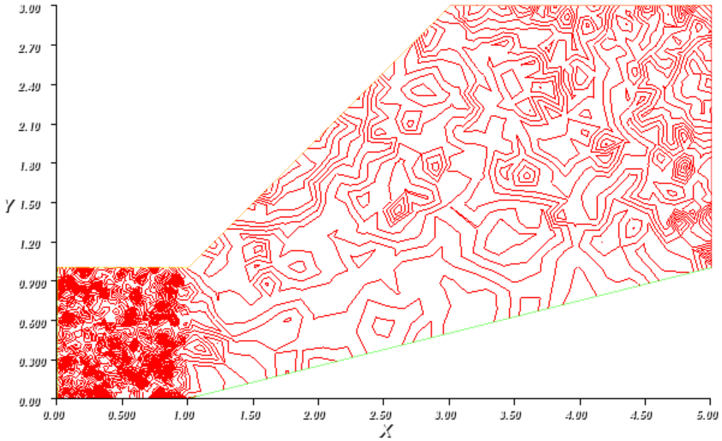
Maximum pressure $\approx 10^5$



WARNING about Dirichlet b.c.

- Dirichlet eigenvalues imposed with large diagonal entry!
- Other matrix entries remain nonzero
- Put **`on(a4, u1=0)`** on both **`OP`** and **`B`**, spurious nonzero eigenvalues
- Put **`on(a4, u1=0)`** on only **`OP`** spurious nonzero eigenvalues become 0.

`example23.edp` spurious eigenvector



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Exercise 25

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Section 3.4.2 Radiation boundary condition

Section 3.5 Heated fan blade

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Section 3.9: Chorin's projection for NSE

Exercise 25 (10 points)

The eigenvalues and eigenvectors for Laplace's equation on the unit square are well known. Supposing Dirichlet boundary conditions on the left and bottom sides of the square, and Neumann conditions on the right and top of the square, the eigenpairs are, for posiitve integers *m* and *n*,

$$\lambda_{m,n} = -\frac{\pi^2}{4}(m^2 + n^2)$$
$$u_{m,n} = \sin(\frac{m\pi x}{2})\sin(\frac{n\pi y}{2}).$$

1. There is additional discussion of the eigenproblem in Section 9.4 of the FreeFem++ book.
2. Approximately confirm the above eigenvalue expression for $m, n = 1, 2$ by computing the eigenvalues using FreeFem++.
3. How many mesh points on each side of the square did you use to get the accuracy you achieved. What accuracy do you observe with twice as many mesn points on each side of the square?
4. Please send me plots of the four eigenvectors you found.
5. Be sure to send me your `.edp` files.

Topics

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Example 24, transient convective cooling

- Cooling of a bimetal plate in 2D

$$\frac{\partial u}{\partial t} - \nabla \cdot \kappa \nabla u = 0 \text{ in } \Omega \times (0, T)$$
$$u(x, y, 0) = u_0 + \frac{x}{L} u_1$$
$$\kappa \frac{\partial u}{\partial n} + \alpha(u - u_e) = 0 \text{ on } \Gamma \times (0, T)$$

- Euler implicit time discretization

$$\int_\Omega \left(\frac{u^n - u^{n-1}}{\Delta t} w + \kappa \nabla u^n \cdot \nabla w \right) dx + \int_\Gamma \alpha(u^n - u_e) w \, ds = 0$$

- $\kappa = 0.2$ on the lower half, 2.0 on the upper half

example24.edp code

```
func u0 = 10 + 90*x/6;  // initial temperatur distribution
func k = 1.8 * (y<0.5) + 0.2; // 2 values of k

real ue = 25, alpha = 0.25, T = 5, dt = 0.1 ;

mesh Th=square(30,5,[6*x, y]);
// Bottom=1, right=2, top=3, left=4 by default
```

example24.edp code

```
func u0 = 10 + 90*x/6;  // initial temperatur distribution
func k = 1.8 * (y<0.5) + 0.2; // 2 values of k

real ue = 25, alpha = 0.25, T = 5, dt = 0.1 ;

mesh Th=square(30,5,[6*x, y]);
// Bottom=1, right=2, top=3, left=4 by default

fespace Vh(Th,P1);
Vh u=u0, v, uold;
```

example24.edp code

```
func u0 = 10 + 90*x/6;  // initial temperatur distribution
func k = 1.8 * (y<0.5) + 0.2; // 2 values of k

real ue = 25, alpha = 0.25, T = 5, dt = 0.1 ;

mesh Th=square(30,5,[6*x, y]);
// Bottom=1, right=2, top=3, left=4 by default

fespace Vh(Th,P1);
Vh u=u0, v, uold;

// for the flat plate
problem thermic(u,v)= int2d(Th)(u*v/dt + k*(dx(u) * dx(v) + dy(u) * dy(v)))
                + int1d(Th,1,3)(alpha*u*v)
                - int1d(Th,1,3)(alpha*ue*v)
                - int2d(Th)(uold*v/dt) + on(2,4,u=u0);
```

example24.edp code

```
func u0 = 10 + 90*x/6;  // initial temperatur distribution
func k = 1.8 * (y<0.5) + 0.2; // 2 values of k

real ue = 25, alpha = 0.25, T = 5, dt = 0.1 ;

mesh Th=square(30,5,[6*x, y]);
// Bottom=1, right=2, top=3, left=4 by default

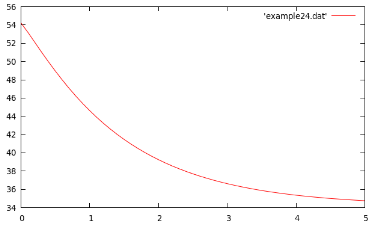
fespace Vh(Th,P1);
Vh u=u0, v, uold;

// for the flat plate
problem thermic(u,v)= int2d(Th)(u*v/dt + k*(dx(u) * dx(v) + dy(u) * dy(v)))
                + int1d(Th,1,3)(alpha*u*v)
                - int1d(Th,1,3)(alpha*ue*v)
                - int2d(Th)(uold*v/dt) + on(2,4,u=u0);

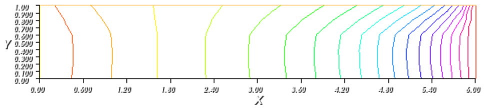
// write a 2-column file
ofstream ff("example24.dat");
for(real t=0; t < T + dt/100.; t += dt){
    uold=u;
    thermic;
    ff << t << " " << u(3,0.5) << endl;
    plot(u);
}
```

Example 24 results

Temperature *vs.* time



Temperature contours



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- Section 3.8: Stokes flow
- Section 3.9: Chorin’s projection for NSE

Example 25, nonlinear boundary condition

- Change boundary conditions for radiation

$$\kappa \frac{\partial u}{\partial n} + \alpha(u - u_e) + c\left((u+273)^4 - (u_e+273)^4\right) = 0$$

- Solution by Picard iteration (successive substitution) after employing the identity $a^4 - b^4 = (a-b)(a+b)(a^2+b^2)$

$$\kappa \frac{\partial u^m}{\partial n} + \alpha(u^m - u_e) + c(u^m - e_e)(u^{m-1} + u_e + 546) \\ \left((u^{m-1}+273)^2 + (u_e+273)^2\right) = 0$$

example25.edp code

```
verbosity = 0;
func u0 = 10 + 90*x/6;
func k = 1.8 * (y<0.5) + 0.2;

real ue = 25, alpha = 0.25, T = 5, dt = 0.1 ;

mesh Th=square(30,5,[6*x,y]);
fespace Vh(Th,P1);
Vh u=u0, uold;

real rad=1e-8, uek=ue+273;
Vh vold, w, v=u0-ue, b;
// v is (u-ue)
problem thermradia(v,w)
    = int2d(Th)(v*w/dt + k*(dx(v) * dx(w) + dy(v) * dy(w)))
                + int1d(Th,1,3)(b*v*w)
                - int2d(Th)(vold*w/dt) + on(2,4,v=u0-ue);
```

example25.edp code

```
verbosity = 0;
func u0 = 10 + 90*x/6;
func k = 1.8 * (y<0.5) + 0.2;

real ue = 25, alpha = 0.25, T = 5, dt = 0.1 ;

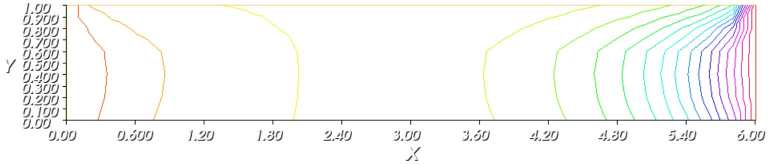
mesh Th=square(30,5,[6*x,y]);
fespace Vh(Th,P1);
Vh u=u0, uold;

real rad=1e-8, uek=ue+273;
Vh vold, w, v=u0-ue, b;
// v is (u-ue)
problem thermradia(v,w)
    = int2d(Th)(v*w/dt + k*(dx(v) * dx(w) + dy(v) * dy(w)))
                + int1d(Th,1,3)(b*v*w)
                - int2d(Th)(vold*w/dt) + on(2,4,v=u0-ue);

for(real t=0;t<T;t+=dt){
    vold=v;
    // 5 Picard iterations
    for(int m=0; m<5; m++){
       b= alpha + rad * (v + 2*uek) * ((v+uek)^2 + uek^2);
       thermradia;
    }
}
vold=v+ue;
plot(vold);
interactive();
```

`example25.edp` results

Solution with radiation boundary condition



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Example 26, Heat convection around an airfoil

- Potential flow (incompressible, irrotational flow)
- $\nabla \cdot u = 0$ and $\nabla \times u = 0$
- So there is ψ satisfying $\Delta\psi = 0$
- And $u = (u_x, u_y)$ where $u_x = \partial\psi/\partial y$ and $u_y = -\partial\psi/\partial x$.
- $\psi =$constant at walls since $\partial u/\partial n = 0$
- Prescribed velocity at a boundary translates into nonconstant Dirichlet data for ψ

Flow around an airfoil

- "Infinity" is a circle, C, of radius 5
- Airfoil, S, is NACA0012, given by

$$y = 0.17735\sqrt{x} - 0.075597x - 0.212836x^2 + 0.17363x^3 - 0.06254x^4$$

- Equations

$$\Delta\psi = 0 \text{ in } \Omega, \qquad \psi|_S = 0, \qquad \psi_C = u_\infty y,$$

and $\partial\Omega = C \cup S$.

example26.edp code

```
real S=99;
border C(t=0,2*pi) {  x=3*cos(t);  y=3*sin(t);}
```

example26.edp code

```
real S=99;
border C(t=0,2*pi) {  x=3*cos(t);  y=3*sin(t);}

border Splus(t=0,1){  x = t; y = 0.17735*sqrt(t) - 0.075597*t
        - 0.212836*(t^2) + 0.17363*(t^3) - 0.06254*(t^4); label=S;}
border Sminus(t=1,0){  x =t; y= -(0.17735*sqrt(t) - 0.075597*t
        - 0.212836*(t^2) + 0.17363*(t^3) - 0.06254*(t^4)); label=S;}
```

example26.edp code

```
real S=99;
border C(t=0,2*pi) {  x=3*cos(t);  y=3*sin(t);}

border Splus(t=0,1){  x = t; y = 0.17735*sqrt(t) - 0.075597*t
        - 0.212836*(t^2) + 0.17363*(t^3) - 0.06254*(t^4); label=S;}
border Sminus(t=1,0){  x =t; y= -(0.17735*sqrt(t) - 0.075597*t
        - 0.212836*(t^2) + 0.17363*(t^3) - 0.06254*(t^4)); label=S;}

mesh Th= buildmesh(C(50)+Splus(70)+Sminus(70));
```

example26.edp code

```
real S=99;
border C(t=0,2*pi) {  x=3*cos(t);  y=3*sin(t);}

border Splus(t=0,1){  x = t; y = 0.17735*sqrt(t) - 0.075597*t
        - 0.212836*(t^2) + 0.17363*(t^3) - 0.06254*(t^4); label=S;}
border Sminus(t=1,0){  x =t; y= -(0.17735*sqrt(t) - 0.075597*t
        - 0.212836*(t^2) + 0.17363*(t^3) - 0.06254*(t^4)); label=S;}

mesh Th= buildmesh(C(50)+Splus(70)+Sminus(70));

fespace Vh(Th,P2);
Vh psi,w;
```

example26.edp code

```
real S=99;
border C(t=0,2*pi) {  x=3*cos(t);  y=3*sin(t);}

border Splus(t=0,1){  x = t; y = 0.17735*sqrt(t) - 0.075597*t
        - 0.212836*(t^2) + 0.17363*(t^3) - 0.06254*(t^4); label=S;}
border Sminus(t=1,0){  x =t; y= -(0.17735*sqrt(t) - 0.075597*t
        - 0.212836*(t^2) + 0.17363*(t^3) - 0.06254*(t^4)); label=S;}

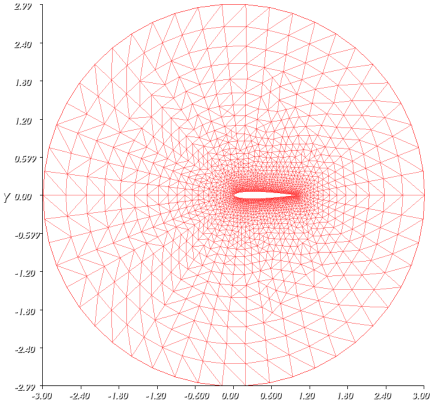
mesh Th= buildmesh(C(50)+Splus(70)+Sminus(70));

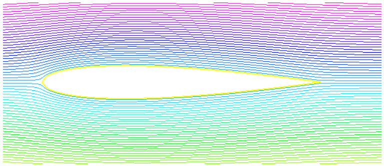
fespace Vh(Th,P2);
Vh psi,w;

solve potential(psi, w) = int2d(Th)(dx(psi)*dx(w) + dy(psi)*dy(w)) +
  on(C, psi=y) + on(S, psi=0);

plot(psi, wait=true);
plot(Th, wait=true);
```

Example 26 flow results





example26.edp code

```
border D(t=0,2){x=1+t; y=0;}
mesh Sh = buildmesh(C(25) + Splus(-90) + Sminus(-90) + D(200));
```

example26.edp code

```
border D(t=0,2){x=1+t; y=0;}
mesh Sh = buildmesh(C(25) + Splus(-90) + Sminus(-90) + D(200));

fespace Wh(Sh, P1);
Wh v,vv;
```

example26.edp code

```
border D(t=0,2){x=1+t; y=0;}
mesh Sh = buildmesh(C(25) + Splus(-90) + Sminus(-90) + D(200));

fespace Wh(Sh, P1);
Wh v,vv;

int steel = Sh(0.5,0).region, air = Sh(-1,0).region;
```

example26.edp code

```
border D(t=0,2){x=1+t; y=0;}
mesh Sh = buildmesh(C(25) + Splus(-90) + Sminus(-90) + D(200));

fespace Wh(Sh, P1);
Wh v,vv;

int steel = Sh(0.5,0).region, air = Sh(-1,0).region;

fespace W0(Sh,P0);
W0 k = 0.01*(region==air) + 0.1*(region==steel);
W0 u1 = dy(psi)*(region==air), u2 = -dx(psi)*(region==air);
Wh vold = 120*(region==steel);
```

example26.edp code

```
border D(t=0,2){x=1+t; y=0;}
mesh Sh = buildmesh(C(25) + Splus(-90) + Sminus(-90) + D(200));

fespace Wh(Sh, P1);
Wh v,vv;

int steel = Sh(0.5,0).region, air = Sh(-1,0).region;

fespace W0(Sh,P0);
W0 k = 0.01*(region==air) + 0.1*(region==steel);
W0 u1 = dy(psi)*(region==air), u2 = -dx(psi)*(region==air);
Wh vold = 120*(region==steel);

real dt=0.05, nbT=50;
bool factoredMatrix = false;
problem thermic(v, vv, init=factoredMatrix, solver=LU)=
     int2d(Sh)(    v*vv/dt + k*( dx(v)*dx(vv) + dy(v)*dy(vv) )
                 + 10*(u1*dx(v) + u2*dy(v))*vv    )
   - int2d(Sh)(vold*vv/dt);
Velocity=10X, no upwinding
```

example26.edp code

```
border D(t=0,2){x=1+t; y=0;}
mesh Sh = buildmesh(C(25) + Splus(-90) + Sminus(-90) + D(200));

fespace Wh(Sh, P1);
Wh v,vv;

int steel = Sh(0.5,0).region, air = Sh(-1,0).region;

fespace W0(Sh,P0);
W0 k = 0.01*(region==air) + 0.1*(region==steel);
W0 u1 = dy(psi)*(region==air), u2 = -dx(psi)*(region==air);
Wh vold = 120*(region==steel);

real dt=0.05, nbT=50;
bool factoredMatrix = false;
problem thermic(v, vv, init=factoredMatrix, solver=LU)=
     int2d(Sh)(     v*vv/dt + k*( dx(v)*dx(vv) + dy(v)*dy(vv) )
                  + 10*(u1*dx(v) + u2*dy(v))*vv     )
   - int2d(Sh)(vold*vv/dt);

for(int i=0;i<nbT;i++){
    v=vold;
    thermic;
    factoredMatrix = true;
}
```

example26.edp code

```
border D(t=0,2){x=1+t; y=0;}
mesh Sh = buildmesh(C(25) + Splus(-90) + Sminus(-90) + D(200));

fespace Wh(Sh, P1);
Wh v,vv;

int steel = Sh(0.5,0).region, air = Sh(-1,0).region;

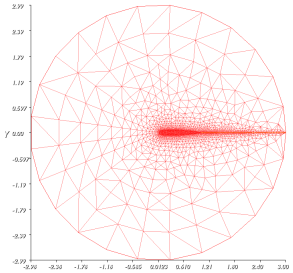
fespace W0(Sh,P0);
W0 k = 0.01*(region==air) + 0.1*(region==steel);
W0 u1 = dy(psi)*(region==air), u2 = -dx(psi)*(region==air);
Wh vold = 120*(region==steel);

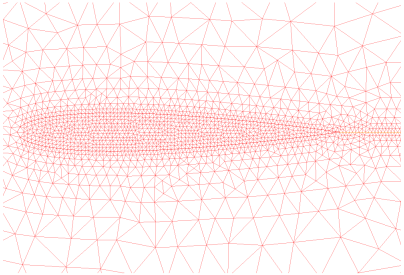
real dt=0.05, nbT=50;
bool factoredMatrix = false;
problem thermic(v, vv, init=factoredMatrix, solver=LU)=
     int2d(Sh)(     v*vv/dt + k*( dx(v)*dx(vv) + dy(v)*dy(vv) )
                  + 10*(u1*dx(v) + u2*dy(v))*vv     )
   - int2d(Sh)(vold*vv/dt);

for(int i=0;i<nbT;i++){
    v=vold;
    thermic;
    factoredMatrix = true;
}

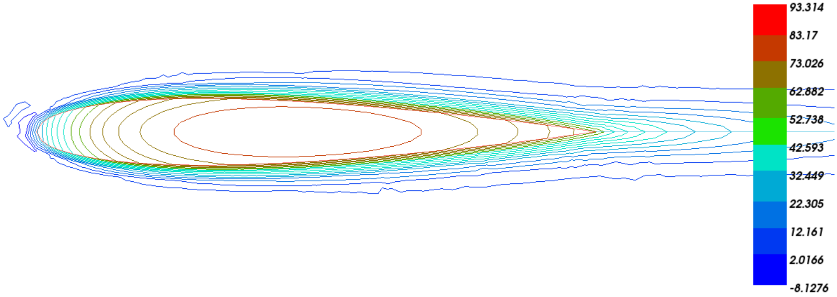
plot(v,wait=1);
plot(Th,wait=1);
plot(Sh,wait=1);
```

Example 26 mesh





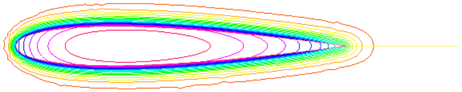
Example 26 thermal results



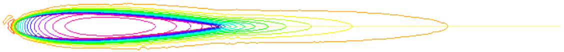
Upwind differencing?

- Upwind differencing is “required” when convective effects become much larger than diffusive effects.
- Velocity was multiplied by 10 to make convective effects “visible.”

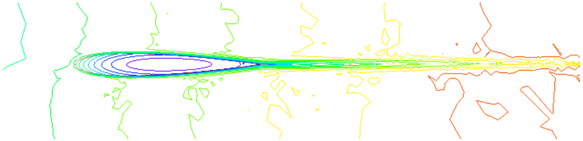
Effect of increasing velocity



Velocity $\times 1$



Velocity $\times 10$



Velocity $\times 100$

Mesh control

- Mesh near “trailing edge” is a problem
    - Singularity in Potential flow solution
- Example 16 (shedding using FEniCS): mesh refinement
- Above, inserted line with many points
- Could insert regions with given number points on boundaries

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Exercise 25

Section 3.4 Thermal convection

Section 3.4.2 Radiation boundary condition

Section 3.5 Heated fan blade

Section 3.6: Rotating hill

Syntax of macros

Section 3.7: Elasticity

Section 3.8: Stokes flow

Section 3.9: Chorin’s projection for NSE

Rotating hill

- Ω is unit disk, $\|x\| \leq 1$
- Fixed velocity field $(u_1, u_2) = (y, -x)$
- Pure convection
$$\frac{\partial c}{\partial t} + u \cdot \nabla c = 0$$
- $c(x, 0) = c^0(x)$
- Exact solution $c(x, t) = R(t)c^0(x, 0)$, where $R(t)$ represents a rotation by $\theta = -t$
- **`example27.edp`** uses **`convect`**.
- $c^0(x)$ is a Gaussian

example27.edp code

```
border C(t=0, 2*pi) { x=cos(t);  y=sin(t); };
mesh Th = buildmesh(C(100));

fespace Uh(Th,P1);
Uh cold, c = exp(-10*((x-0.3)^2 +(y-0.3)^2));

real dt = 0.17,t=0;
Uh u1 = y, u2 = -x;
```

example27.edp code

```
border C(t=0, 2*pi) { x=cos(t);  y=sin(t); };
mesh Th = buildmesh(C(100));

fespace Uh(Th,P1);
Uh cold, c = exp(-10*((x-0.3)^2 +(y-0.3)^2));

real dt = 0.17,t=0;
Uh u1 = y, u2 = -x;

real [int] viso=[-0.1, 0, 0.5, 0.1, 0.5, 0.2, 0.25, 0.3, 0.35, 0.4,
                 0.45, 0.5, 0.55, 0.6, 0.65, 0.7, 0.75, 0.8, 0.9, 1];
```

example27.edp code

```
border C(t=0, 2*pi) { x=cos(t);  y=sin(t); };
mesh Th = buildmesh(C(100));

fespace Uh(Th,P1);
Uh cold, c = exp(-10*((x-0.3)^2 +(y-0.3)^2));

real dt = 0.17,t=0;
Uh u1 = y, u2 = -x;




for (int m=0; m<2*pi/dt ; m++) {
    t += dt;
    cold=c;
    c=convect([u1,u2], -dt, cold);
    plot(c,cmm=" t="+t + ", min=" + c[].min + ", max=" +  c[].max,
         viso=viso, dim=3, fill=true);
}
plot(c,wait=true,fill=false,value=true,viso=viso,
  cmm="example27, min=" + c[].min + ", max=" +  c[].max);
```

example27.edp code

```
border C(t=0, 2*pi) { x=cos(t);  y=sin(t); };
mesh Th = buildmesh(C(100));

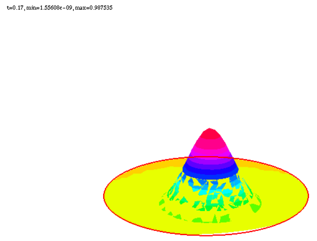
fespace Uh(Th,P1);
Uh cold, c = exp(-10*((x-0.3)^2 +(y-0.3)^2));

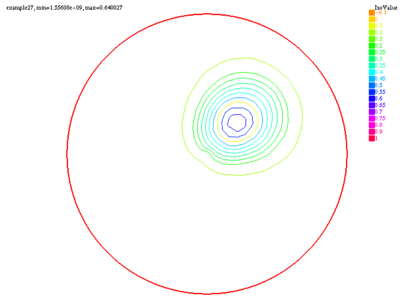
real dt = 0.17,t=0;
Uh u1 = y, u2 = -x;




for (int m=0; m<2*pi/dt ; m++) {
    t += dt;
    cold=c;
    c=convect([u1,u2], -dt, cold);
    plot(c,cmm=" t="+t + ", min=" + c[].min + ", max=" +  c[].max,
         viso=viso, dim=3, fill=true);
}
plot(c,wait=true,fill=false,value=true,viso=viso,
  cmm="example27, min=" + c[].min + ", max=" +  c[].max);
```

`example27.edp` output





Example 27 uses discontinuous Galerkin

This is the so-called dual-P_1^{DC} formulation (Ref. Ern[11])

$$\int_\Omega \frac{c^n - c^{n-1}}{\Delta t}\, dx + \int_E (\alpha |n \cdot u| - 0.5 n \cdot u [c] w\, ds = \int_{e_\Gamma^-} |n \cdot u| c w\, ds$$

- E is the set of inner edges
- E^- is the set of boundary edges with $u \cdot n < 0$ (none here)
- $[c]$ is jump $(c^+ - c^-)$ where $+$ refers to the right of the oriented edge

example28.edp code

```
// example28.edp, from file chapt3/convects.edp
// With Discontinuous Galerkin
border C(t=0, 2*pi) { x=cos(t);  y=sin(t); };
mesh Th = buildmesh(C(100));
fespace Vh(Th,P1dc);

Vh w, ccold, v1 = y, v2 = -x, cc = exp(-10*( (x-0.3)^2 + (y-0.3)^2) );
real u, alpha=0.5,  dt = 0.05;
```

example28.edp code

```
// example28.edp, from file chapt3/convects.edp
// With Discontinuous Galerkin
border C(t=0, 2*pi) { x=cos(t);  y=sin(t); };
mesh Th = buildmesh(C(100));
fespace Vh(Th,P1dc);

Vh w, ccold, v1 = y, v2 = -x, cc = exp(-10*( (x-0.3)^2 + (y-0.3)^2) );
real u, alpha=0.5,  dt = 0.05;

macro n()(N.x*v1 + N.y*v2) //
```

example28.edp code

```
// example28.edp, from file chapt3/convects.edp
// With Discontinuous Galerkin
border C(t=0, 2*pi) { x=cos(t);  y=sin(t); };
mesh Th = buildmesh(C(100));
fespace Vh(Th,P1dc);

Vh w, ccold, v1 = y, v2 = -x, cc = exp(-10*( (x-0.3)^2 + (y-0.3)^2) );
real u, alpha=0.5,  dt = 0.05;

macro n()(N.x*v1 + N.y*v2) //

problem  Adual(cc, w) = int2d(Th)( (cc/dt + (v1*dx(cc) + v2*dy(cc)) )*w )
  + intalledges(Th)( (1-nTonEdge)*w*( alpha*abs(n) - n/2 )*jump(cc) )

  - int2d(Th)( ccold*w/dt );
```

example28.edp code

```
// example28.edp, from file chapt3/convects.edp
// With Discontinuous Galerkin
border C(t=0, 2*pi) { x=cos(t);  y=sin(t); };
mesh Th = buildmesh(C(100));
fespace Vh(Th,P1dc);

Vh w, ccold, v1 = y, v2 = -x, cc = exp(-10*( (x-0.3)^2 + (y-0.3)^2) );
real u, alpha=0.5,  dt = 0.05;

macro n()(N.x*v1 + N.y*v2) //

problem  Adual(cc, w) = int2d(Th)( (cc/dt + (v1*dx(cc) + v2*dy(cc)) )*w )
  + intalledges(Th)( (1-nTonEdge)*w*( alpha*abs(n) - n/2 )*jump(cc) )
//  - int1d(Th,C)( (n(u)<0)*abs(n(u) )*cc*w)  // unused: cc=0 on boundary
  - int2d(Th)( ccold*w/dt );
```

example28.edp code

```
// example28.edp, from file chapt3/convects.edp
// With Discontinuous Galerkin
border C(t=0, 2*pi) { x=cos(t);  y=sin(t); };
mesh Th = buildmesh(C(100));
fespace Vh(Th,P1dc);

Vh w, ccold, v1 = y, v2 = -x, cc = exp(-10*( (x-0.3)^2 + (y-0.3)^2) );
real u, alpha=0.5,  dt = 0.05;

macro n()(N.x*v1 + N.y*v2) //

problem  Adual(cc, w) = int2d(Th)( (cc/dt + (v1*dx(cc) + v2*dy(cc)) )*w )
  + intalledges(Th)( (1-nTonEdge)*w*( alpha*abs(n) - n/2 )*jump(cc) )
//  - int1d(Th,C)( (n(u)<0)*abs(n(u) )*cc*w)  // unused: cc=0 on boundary
  - int2d(Th)( ccold*w/dt );

real [int] viso=[-0.1, 0, 0.5, 0.1, 0.5, 0.2, 0.25, 0.3, 0.35, 0.4,
                 0.45, 0.5, 0.55, 0.6, 0.65, 0.7, 0.75, 0.8, 0.9, 1];
```

example28.edp code

```
// example28.edp, from file chapt3/convects.edp
// With Discontinuous Galerkin
border C(t=0, 2*pi) { x=cos(t);  y=sin(t); };
mesh Th = buildmesh(C(100));
fespace Vh(Th,P1dc);

Vh w, ccold, v1 = y, v2 = -x, cc = exp(-10*( (x-0.3)^2 + (y-0.3)^2) );
real u, alpha=0.5,  dt = 0.05;

macro n()(N.x*v1 + N.y*v2) //

problem  Adual(cc, w) = int2d(Th)( (cc/dt + (v1*dx(cc) + v2*dy(cc)) )*w )
  + intalledges(Th)( (1-nTonEdge)*w*( alpha*abs(n) - n/2 )*jump(cc) )
//  - int1d(Th,C)( (n(u)<0)*abs(n(u) )*cc*w)  // unused: cc=0 on boundary
  - int2d(Th)( ccold*w/dt );

real [int] viso=[-0.1, 0, 0.5, 0.1, 0.5, 0.2, 0.25, 0.3, 0.35, 0.4,
                 0.45, 0.5, 0.55, 0.6, 0.65, 0.7, 0.75, 0.8, 0.9, 1];

for (real t=0; t< 2*pi ; t+=dt){
  ccold=cc;
  Adual;
  plot(cc, fill=true, dim=3, viso=viso,
   cmm="t="+t + ", min=" + cc[].min + ", max=" +  cc[].max);
};

plot(cc,wait=true,fill=false,value=true,viso=viso,
  cmm="example28, min=" + cc[].min + ", max=" +  cc[].max);
```

Some details of new functions

The integral over *interior* edges is given by

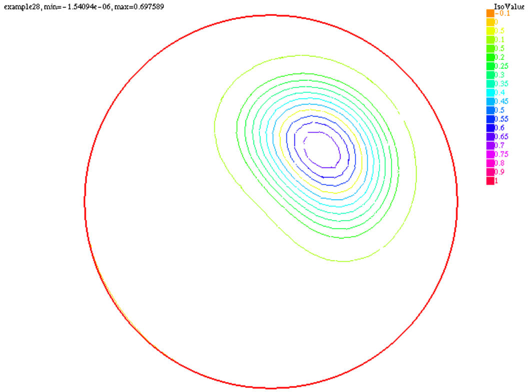
$$\int_E (\alpha|n \cdot u| - 0.5 n \cdot u[c] w \, ds$$

And is implemented as

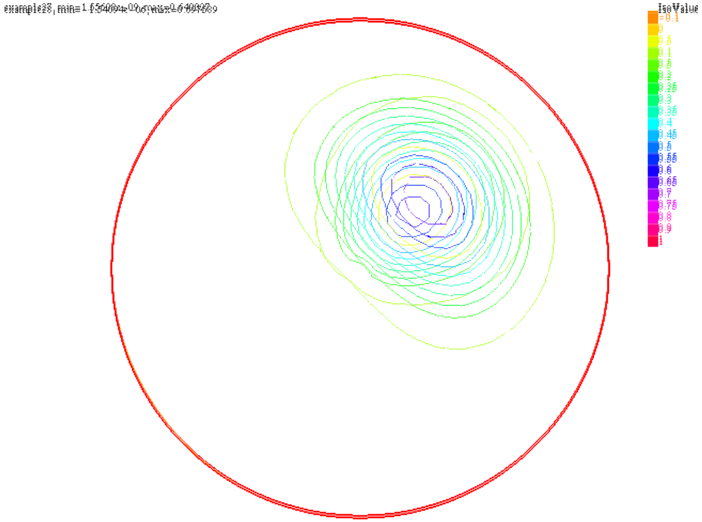
```
 intalledges(Th)( (1-nTonEdge)*w*( al*abs(n) - n/2 )*jum
))
```

- **`intalledges`** is an integral over *all* edges, including boundary edges
- **`nTonEdge`** is the number of triangles meeting at this edge. In a 2d geometry, this can only be 1 or 2.
- **`(1-nTonEdge)`** is (-1) for interior edges, 0 for boundary edges.
- **`jump(cc)`** computes $cc^- - cc^+$
- $[c] = c^+ - c^-$ as used by Ern & Guermond, *Theory and practice of finite elements*, referenced in the FreeFem++ book.
- The explanation on p. 35 is slightly misleading.

example28.edp final plot



Comparison of 27 and 28



Comparison of 27 and 28

27 declined to 0.64, 28 declined to .70

example29.edp: faster

```
varf aadual(cc, w) = int2d(Th)( (cc/dt + (v1*dx(cc) + v2*dy(cc)) )*w )
  + intalledges(Th)( (1-nTonEdge)*w*( alpha*abs(n) - n/2 )*jump(cc) );
varf bbdual(ccold, w) =  - int2d(Th)( ccold*w/dt);
```

example29.edp: faster

```
varf aadual(cc, w) = int2d(Th)( (cc/dt + (v1*dx(cc) + v2*dy(cc)) )*w )
  + intalledges(Th)( (1-nTonEdge)*w*( alpha*abs(n) - n/2 )*jump(cc) );
varf bbdual(ccold, w) =  - int2d(Th)( ccold*w/dt);

bool reuseMatrix = false;
matrix AA = aadual(Vh,Vh);
matrix BB = bbdual(Vh,Vh);
set (AA, init=reuseMatrix, solver=UMFPACK);
Vh rhs=0;
```

example29.edp: faster

```
varf aadual(cc, w) = int2d(Th)( (cc/dt + (v1*dx(cc) + v2*dy(cc)) )*w )
  + intalledges(Th)( (1-nTonEdge)*w*( alpha*abs(n) - n/2 )*jump(cc) );
varf bbdual(ccold, w) =  - int2d(Th)( ccold*w/dt);

bool reuseMatrix = false;
matrix AA = aadual(Vh,Vh);
matrix BB = bbdual(Vh,Vh);
set (AA, init=reuseMatrix, solver=UMFPACK);
Vh rhs=0;

for (real t=0; t< 2*pi ; t+=dt) {
  ccold = cc;
  rhs[] = BB * ccold[];
  cc[]  = AA^-1 * rhs[];
  reuseMatrix = true;
  plot(cc, fill=true, dim=3, viso=viso,
    cmm="t="+t + ", min=" + cc[].min + ", max=" +  cc[].max);
};
```

Exercise 26 (10 points)

Consider the three forms of the rotating hill problem, Examples 27, 28, and 29.

1. Run each to a limit of 16π instead of 2π. Send me a copy of the final plot from each case. What would you conclude from your experiment, if anything?
2. Return to Example 29 and revise it so that it uses ordinary P1 elements and the usual finite element discretization of convection. Run it twice, once until a limit of 2π, and once to a limit of 16π. What would you conclude from this experiment?

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Macros

- Macros can be defined either with parameters:

```
macro <identifier>(<parameter list>) <replacement token list> // EOM
```

- Or without parameters

```
macro <identifier>()  <replacement token list> // EOM
```

- Macros *must* end with the comment characters `//`.

Examples of macros

```
macro n() (N.x*v1 + N.y*v2)  // EOM

macro div(u,v) (dx(u) + dy(v)) // EOM

macro epsilon(u1,u2) [ dx(u1), dy(u2), (dy(u1)+dx(u2)) ] //EOM

macro Grad(u1,u2) [ dx(u1), dy(u1), dx(u2), dy(u2) ]    // EOM

macro UgradV(u1,u2,v1,v2) [ [u1,u2]'*[dx(v1),dy(v1)] ,
                            [u1,u2]'*[dx(v2),dy(v2)] ]  // EOM
```

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Elasticity

The equations governing elastic (small) deformation can be written

$$\begin{aligned}
-\nabla \cdot \sigma &= t \text{ in } \Omega \\
\sigma &= \lambda(\nabla \cdot u)I + 2\mu\epsilon \\
\epsilon &= \frac{1}{2}(\nabla u + \nabla u^T) \\
\epsilon_{ij} &= \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)
\end{aligned}$$

where σ is a stress tensor and ϵ is a deformation tensor. The deformation u is a vector. In 2D, both u and x have 2 components. The variational form becomes

$$\int_\Omega \lambda\nabla \cdot u\nabla \cdot v + 2\mu\epsilon(u) : \epsilon(v) - \int_\Omega vf = 0$$

example30.edp code

```
verbosity = 0;
mesh Th=square(10, 10, [20*x, 2*y-1]);
fespace Vh(Th, P2);
Vh u, v, uu, vv;
```

example30.edp code

```
verbosity = 0;
mesh Th=square(10, 10, [20*x, 2*y-1]);
fespace Vh(Th, P2);
Vh u, v, uu, vv;

real sqrt2 = sqrt(2.);
macro epsilon(u1,u2)  [dx(u1),dy(u2),(dy(u1) + dx(u2))/sqrt2] // EOM
macro div(u,v) ( dx(u) + dy(v) ) // EOM
```

example30.edp code

```
verbosity = 0;
mesh Th=square(10, 10, [20*x, 2*y-1]);
fespace Vh(Th, P2);
Vh u, v, uu, vv;

real sqrt2 = sqrt(2.);
macro epsilon(u1,u2)  [dx(u1),dy(u2),(dy(u1) + dx(u2))/sqrt2] // EOM
macro div(u,v) ( dx(u) + dy(v) ) // EOM

real E = 21e5, nu = 0.28, mu= E/(2*(1 + nu));
real lambda = E * nu / ((1+nu) * (1-2*nu)), f = -1;
```

example30.edp code

```
verbosity = 0;
mesh Th=square(10, 10, [20*x, 2*y-1]);
fespace Vh(Th, P2);
Vh u, v, uu, vv;

real sqrt2 = sqrt(2.);
macro epsilon(u1,u2)  [dx(u1),dy(u2),(dy(u1) + dx(u2))/sqrt2] // EOM
macro div(u,v) ( dx(u) + dy(v) ) // EOM

real E = 21e5, nu = 0.28, mu= E/(2*(1 + nu));
real lambda = E * nu / ((1+nu) * (1-2*nu)), f = -1;

solve lame([u, v],[uu, vv])= int2d(Th)(
        lambda * div(u, v) * div(uu, vv)
        +2.*mu*( epsilon(u, v)' * epsilon(uu, vv) ) )
        - int2d(Th)( f*vv )
        + on(4, u=0, v=0);
```

example30.edp code

```
verbosity = 0;
mesh Th=square(10, 10, [20*x, 2*y-1]);
fespace Vh(Th, P2);
Vh u, v, uu, vv;

real sqrt2 = sqrt(2.);
macro epsilon(u1,u2)  [dx(u1),dy(u2),(dy(u1) + dx(u2))/sqrt2] // EOM
macro div(u,v) ( dx(u) + dy(v) ) // EOM

real E = 21e5, nu = 0.28, mu= E/(2*(1 + nu));
real lambda = E * nu / ((1+nu) * (1-2*nu)), f = -1;

solve lame([u, v],[uu, vv])= int2d(Th)(
        lambda * div(u, v) * div(uu, vv)
        +2.*mu*( epsilon(u, v)' * epsilon(uu, vv) ) )
        - int2d(Th)( f*vv )
        + on(4, u=0, v=0);

real coef=100;
plot([u,v], wait=true, coef=coef);

mesh th1 = movemesh(Th, [x + u*coef, y + v*coef]);
plot(th1, wait=true);
```

example30.edp code

```
verbosity = 0;
mesh Th=square(10, 10, [20*x, 2*y-1]);
fespace Vh(Th, P2);
Vh u, v, uu, vv;

real sqrt2 = sqrt(2.);
macro epsilon(u1,u2)  [dx(u1),dy(u2),(dy(u1) + dx(u2))/sqrt2] // EOM
macro div(u,v) ( dx(u) + dy(v) ) // EOM

real E = 21e5, nu = 0.28, mu= E/(2*(1 + nu));
real lambda = E * nu / ((1+nu) * (1-2*nu)), f = -1;

solve lame([u, v],[uu, vv])= int2d(Th)(
        lambda * div(u, v) * div(uu, vv)
        +2.*mu*( epsilon(u, v)' * epsilon(uu, vv) ) )
        - int2d(Th)( f*vv )
        + on(4, u=0, v=0);

real coef=100;
plot([u,v], wait=true, coef=coef);

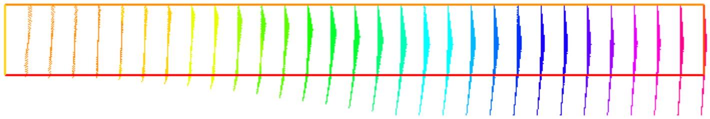
mesh th1 = movemesh(Th, [x + u*coef, y + v*coef]);
plot(th1, wait=true);

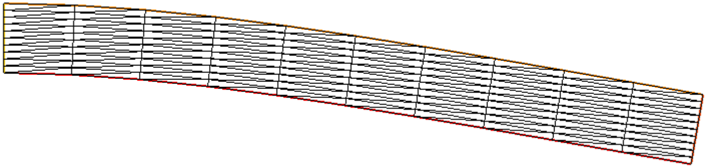
real dxmin  = u[].min;
real dymin  = v[].min;

cout << " - dep.  max   x = "<< dxmin<< " y=" << dymin << endl;
cout << "   dep.  (20,0)  = " << u(20,0) << " " << v(20,0) << endl;
```

`example30.edp` results

Deformation of a beam under its own weight





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Example 31, Driven cavity for Stokes flow

- Extremely slow incompressible flow: Stokes

$$\begin{aligned} -\Delta u + \nabla p &= 0 \\ \nabla \cdot u &= 0 \end{aligned}$$

- $u = (u_1, u_2)$ is velocity, p is a scalar pressure.
- Square with lid moving to the right
- Dirichlet conditions on velocity
- Pressure needs to eliminate the constant null eigenvector

example31.edp code

```
verbose = 0;
int n = 3;
mesh Th=square(10*n, 10*n);
```

example31.edp code

```
verbose = 0;
int n = 3;
mesh Th=square(10*n, 10*n);

fespace Uh(Th, P1b);
Uh u,v,uu,vv;
fespace Ph(Th, P1);
Ph p,pp;
```

example31.edp code

```
verbose = 0;
int n = 3;
mesh Th=square(10*n, 10*n);

fespace Uh(Th, P1b);
Uh u,v,uu,vv;
fespace Ph(Th, P1);
Ph p,pp;

solve stokes([u, v, p],[uu, vv, pp]) =
    int2d(Th)(dx(u)*dx(uu) + dy(u)*dy(uu) + dx(v)*dx(vv) + dy(v)*dy(vv)
            + dx(p)*uu + dy(p)*vv + pp*(dx(u)+dy(v))
            -1e-10*p*pp)
            + on(1,2,4, u=0, v=0) + on(3, u=1, v=0);
```

example31.edp code

```
verbose = 0;
int n = 3;
mesh Th=square(10*n, 10*n);

fespace Uh(Th, P1b);
Uh u,v,uu,vv;
fespace Ph(Th, P1);
Ph p,pp;

solve stokes([u, v, p],[uu, vv, pp]) =
    int2d(Th)(dx(u)*dx(uu) + dy(u)*dy(uu) + dx(v)*dx(vv) + dy(v)*dy(vv)
            + dx(p)*uu + dy(p)*vv + pp*(dx(u)+dy(v))
            -1e-10*p*pp)
            + on(1,2,4, u=0, v=0) + on(3, u=1, v=0);
```

example31.edp code

```
verbose = 0;
int n = 3;
mesh Th=square(10*n, 10*n);

fespace Uh(Th, P1b);
Uh u,v,uu,vv;
fespace Ph(Th, P1);
Ph p,pp;

solve stokes([u, v, p],[uu, vv, pp]) =
    int2d(Th)(dx(u)*dx(uu) + dy(u)*dy(uu) + dx(v)*dx(vv) + dy(v)*dy(vv)
            + dx(p)*uu + dy(p)*vv + pp*(dx(u)+dy(v))
            -1e-10*p*pp)
            + on(1,2,4, u=0, v=0) + on(3, u=1, v=0);
```

example31.edp code

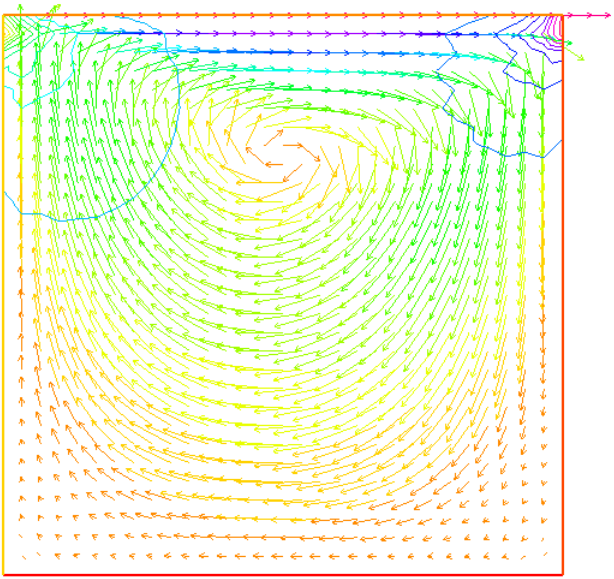
```
verbose = 0;
int n = 3;
mesh Th=square(10*n, 10*n);

fespace Uh(Th, P1b);
Uh u,v,uu,vv;
fespace Ph(Th, P1);
Ph p,pp;

solve stokes([u, v, p],[uu, vv, pp]) =
    int2d(Th)(dx(u)*dx(uu) + dy(u)*dy(uu) + dx(v)*dx(vv) + dy(v)*dy(vv)
            + dx(p)*uu + dy(p)*vv + pp*(dx(u)+dy(v))
            -1e-10*p*pp)
            + on(1,2,4, u=0, v=0) + on(3, u=1, v=0);

plot([u,v],p,wait=true);
```

`example31.edp` results



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Projection algorithm for NSE

- NSE (with Dirichlet boundary conditions)

$$\frac{\partial u}{\partial t} + u \cdot \nabla u + \nabla p - \nu \Delta u = 0$$

- Chorin's algorithm

$$\frac{\tilde{u} - u^{m-1} \circ X^{m-1}}{\Delta t} + \nabla p^{m-1} - \nu \Delta u^{m-1} = 0$$

$$-\Delta p^m = -\nabla \cdot u^{m-1} \circ X^{m-1}, \text{ Neumann b.c.}$$

- Rennacher's improvement

$$-\Delta q = \nabla \cdot u - \overline{\nabla \cdot u}$$

$$u^m = \tilde{u} + \Delta t \nabla q, \qquad p^m = p^{m-1} - q - \overline{p^m - q}$$

Steady flow over a backward-facing step

- Step in time
- Refine mesh periodically
- Stop when step-to-step change is small

example32.edp code

```
border a0(t=1,0){ x=0;              y=t;              label=1;}
border a1(t=0,1){ x=2*t;            y=0;              label=2;}
border a2(t=0,1){ x=2;              y=-t/2;           label=2;}
border a3(t=0,1){ x=2+18*t^1.2;     y=-0.5;           label=2;}
border a4(t=0,1){ x=20;             y=-0.5+1.5*t;     label=3;}
border a5(t=1,0){ x=20*t;           y=1;              label=4;}
int n=1;
mesh Th= buildmesh(a0(3*n) + a1(20*n) + a2(10*n) + a3(150*n) +
                   a4(5*n) + a5(100*n));
plot(Th);
```

example32.edp code

```
border a0(t=1,0){ x=0;            y=t;            label=1;}
border a1(t=0,1){ x=2*t;          y=0;            label=2;}
border a2(t=0,1){ x=2;            y=-t/2;         label=2;}
border a3(t=0,1){ x=2+18*t^1.2;  y=-0.5;         label=2;}
border a4(t=0,1){ x=20;           y=-0.5+1.5*t;  label=3;}
border a5(t=1,0){ x=20*t;         y=1;            label=4;}
int n=1;
mesh Th= buildmesh(a0(3*n) + a1(20*n) + a2(10*n) + a3(150*n) +
                   a4(5*n) + a5(100*n));
plot(Th);

fespace Vh(Th,P1);
```

`example32.edp` code

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fespace Vh(Th,P1);
real nu = 0.0025, dt = 0.2; // Reynolds=200
```

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                   a4(5*n) + a5(100*n));
plot(Th);

fespace Vh(Th,P1);
real nu = 0.0025, dt = 0.2; // Reynolds=200
func uBCin =   4*y*(1-y) * (y>0) * (x<2) ;
func uBCout =  4./1.5*(y+0.5) * (1-y) * (x>19);
```

example32.edp code

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Vh w,u = uBCin, v = 0, p = 0, q = 0;
Vh ubc  = uBCin + uBCout;
```

example32.edp code

```
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plot(Th);

fespace Vh(Th,P1);
real nu = 0.0025, dt = 0.2; // Reynolds=200
func uBCin =   4*y*(1-y) * (y>0) * (x<2) ;
func uBCout =  4./1.5*(y+0.5) * (1-y) * (x>19);
Vh w,u = uBCin, v = 0, p = 0, q = 0;
Vh ubc  = uBCin + uBCout;
real influx0  = int1d(Th,1) (ubc*N.x),
     outflux0 = int1d(Th,3) (ubc*N.x);
real area= int2d(Th)(1.);
```

example32.edp code

```
border a0(t=1,0){ x=0;            y=t;            label=1;}
border a1(t=0,1){ x=2*t;          y=0;            label=2;}
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Vh w,u = uBCin, v = 0, p = 0, q = 0;
Vh ubc  = uBCin + uBCout;
real influx0  = int1d(Th,1) (ubc*N.x),
     outflux0 = int1d(Th,3) (ubc*N.x);
real area= int2d(Th)(1.);

bool reuseMatrix = false;
```

example32.edp code cont'd

```
for(int n=0;n<300;n++){
  Vh uold = u, vold = v, pold = p;
  Vh f=convect( [uold,vold], -dt, uold);
```

example32.edp code cont'd

```
for(int n=0;n<300;n++){
  Vh uold = u, vold = v, pold = p;
  Vh f=convect( [uold,vold], -dt, uold);

  real outflux = int1d(Th, 3) (f*N.x);
  f = f - (influx0 + outflux) / outflux0 * uBCout;
  outflux = int1d(Th, 3) (f*N.x);
  assert( abs( influx0 + outflux ) < 1e-10);
  // WARNING the the output flux must be 0 ...
```

example32.edp code cont'd

```
for(int n=0;n<300;n++){
  Vh uold = u, vold = v, pold = p;
  Vh f=convect( [uold,vold], -dt, uold);

  real outflux = int1d(Th, 3) (f*N.x);
  f = f - (influx0 + outflux) / outflux0 * uBCout;
  outflux = int1d(Th, 3) (f*N.x);
  assert( abs( influx0 + outflux ) < 1e-10);
  // WARNING the the output flux must be 0 ...

  solve pb4u(u, w, init=reuseMatrix, solver=UMFPACK)
        =int2d(Th)(u*w/dt + nu*(dx(u)*dx(w) + dy(u)*dy(w)))
        -int2d(Th)((convect([uold,vold], -dt, uold)/dt - dx(p))*w)
        + on(1,u = 4*y*(1-y)) + on(2,4,u = 0) + on(3,u=f);
  plot(u);
```

example32.edp code cont'd

```
for(int n=0;n<300;n++){
  Vh uold = u, vold = v, pold = p;
  Vh f=convect( [uold,vold], -dt, uold);

  real outflux = int1d(Th, 3) (f*N.x);
  f = f - (influx0 + outflux) / outflux0 * uBCout;
  outflux = int1d(Th, 3) (f*N.x);
  assert( abs( influx0 + outflux ) < 1e-10);
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        -int2d(Th)((convect([uold,vold], -dt, uold)/dt - dx(p))*w)
        + on(1,u = 4*y*(1-y)) + on(2,4,u = 0) + on(3,u=f);
  plot(u);

  solve pb4v(v, w, init=reuseMatrix, solver=UMFPACK)
        = int2d(Th)(v*w/dt + nu*(dx(v)*dx(w) + dy(v)*dy(w)))
        - int2d(Th)((convect([uold,vold], -dt, vold)/dt - dy(p))*w)
        + on(1, 2, 3, 4, v = 0);

  real meandiv = int2d(Th)( dx(u) + dy(v) )/area;
```

example32.edp code cont'd

```
for(int n=0;n<300;n++){
  Vh uold = u, vold = v, pold = p;
  Vh f=convect( [uold,vold], -dt, uold);

  real outflux = int1d(Th, 3) (f*N.x);
  f = f - (influx0 + outflux) / outflux0 * uBCout;
  outflux = int1d(Th, 3) (f*N.x);
  assert( abs( influx0 + outflux ) < 1e-10);
  // WARNING the the output flux must be 0 ...

  solve pb4u(u, w, init=reuseMatrix, solver=UMFPACK)
        =int2d(Th)(u*w/dt + nu*(dx(u)*dx(w) + dy(u)*dy(w)))
        -int2d(Th)((convect([uold,vold], -dt, uold)/dt - dx(p))*w)
        + on(1,u = 4*y*(1-y)) + on(2,4,u = 0) + on(3,u=f);
  plot(u);

  solve pb4v(v, w, init=reuseMatrix, solver=UMFPACK)
        = int2d(Th)(v*w/dt + nu*(dx(v)*dx(w) + dy(v)*dy(w)))
        - int2d(Th)((convect([uold,vold], -dt, vold)/dt - dy(p))*w)
        + on(1, 2, 3, 4, v = 0);

  real meandiv = int2d(Th)( dx(u) + dy(v) )/area;

  solve pb4p(q, w, init=reuseMatrix, solver=UMFPACK)
        = int2d(Th)( dx(q)*dx(w) + dy(q)*dy(w) )
        - int2d(Th)( (dx(u) + dy(v) - meandiv)*w/dt ) + on(3,q=0);

  reuseMatrix = false;
```

example32.edp code cont'd

```
real meanpq = int2d(Th)(pold - q)/area;

if(n%50==49){
   Th = adaptmesh(Th, [u,v], q, err=0.04, nbvx=100000);
   plot(Th, wait=true);
   ubc = uBCin + uBCout; // reinterpolate B.C.
   influx0 = int1d(Th,1) (ubc*N.x);
   outflux0 = int1d(Th,3) (ubc*N.x);
}
```

`example32.edp` code cont'd

```
real meanpq = int2d(Th)(pold - q)/area;

if(n%50==49){
   Th = adaptmesh(Th, [u,v], q, err=0.04, nbvx=100000);
   plot(Th, wait=true);
   ubc = uBCin + uBCout; // reinterpolate B.C.
   influx0 = int1d(Th,1) (ubc*N.x);
   outflux0 = int1d(Th,3) (ubc*N.x);
}

p = pold - q - meanpq;
u = u + dx(q)*dt;
v = v + dy(q)*dt;
```

example32.edp code cont'd

```
    real meanpq = int2d(Th)(pold - q)/area;

    if(n%50==49){
       Th = adaptmesh(Th, [u,v], q, err=0.04, nbvx=100000);
       plot(Th, wait=true);
       ubc = uBCin + uBCout; // reinterpolate B.C.
       influx0 = int1d(Th,1) (ubc*N.x);
       outflux0 = int1d(Th,3) (ubc*N.x);
    }

    p = pold - q - meanpq;
    u = u + dx(q)*dt;
    v = v + dy(q)*dt;

    real err = sqrt(int2d(Th)( square(u - uold) + square(v - vold))/Th.area) ;
    cout << " iter " << n << " Err L2 = " << err << endl;
    if( err < 1e-3 ) break;
  }
```

`example32.edp` code cont'd

```
    real meanpq = int2d(Th)(pold - q)/area;

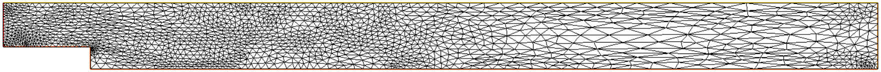
    if(n%50==49){
       Th = adaptmesh(Th, [u,v], q, err=0.04, nbvx=100000);
       plot(Th, wait=true);
       ubc = uBCin + uBCout; // reinterpolate B.C.
       influx0 = int1d(Th,1) (ubc*N.x);
       outflux0 = int1d(Th,3) (ubc*N.x);
    }

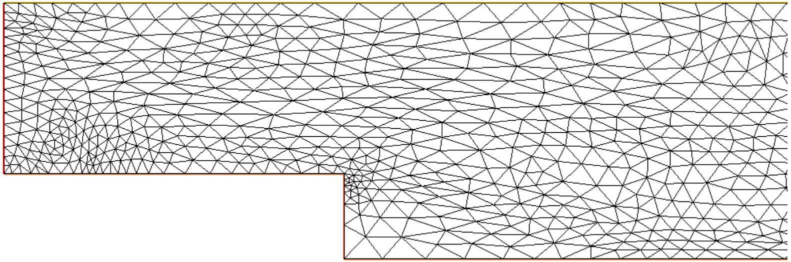
    p = pold - q - meanpq;
    u = u + dx(q)*dt;
    v = v + dy(q)*dt;

    real err = sqrt(int2d(Th)( square(u - uold) + square(v - vold))/Th.area) ;
    cout << " iter " << n << " Err L2 = " << err << endl;
    if( err < 1e-3 ) break;
  }

  plot(Th, wait=true);
  plot(p, wait=true);
  plot(u, wait=true);
```

`example32.edp` final mesh





`example32.edp` final u, p