The Capture of Infinity

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For thousands of years, infinity stood outside the door and was not allowed in, while philosophers and mathematicians argued that infinity was "just an expression" and "not a real concept".

It was only in the 1870's that a single researcher, Georg Cantor, discovered a way to capture the idea of infinity that was mathematically and logically correct.

This talk will try to explain why infinity is such a difficult concept to define, and how Cantor was able to unlock the door and allow infinity to join mathematics.

COUNTING SHEEP

Whether you're trying to fall asleep, or getting your flock ready for market, we all know the importance of counting sheep!

We assume that the first counting was done with the fingers, doing what mathematicians call a **1-to-1 correspondence**.

What Happens When You Need to Go to 11?

$$1, 2, 3, \ldots$$

You can run out of fingers, since you might have 11 sheep to deal with. So very quickly, people came up with a set of idealized fingers, that is, **standard units** that represented the ideas of 1 finger, 2 fingers, and so on.

The number **7** is the property that is shared by 7 dwarves, 7 samurai, and the 7 days of the week.

These abstract quantities are sometimes called counting numbers or **cardinal numbers**, to distinguish them from another idea that involves ordering (1st, 2nd, 3rd...).

Advantages of Using Numbers

Once we have a system of cardinal numbers, we can concentrate on understanding the properties of numbers, instead of sheep.

To understand a flock of sheep, we imagine **matching** each sheep to an imaginary finger. The number of fingers we need is the cardinal number of the flock.

You can't compare apples and oranges, but you *can* compare the number of them.

Axiom of Comparability

Any two numbers a and b are **comparable**:
either $a \leq b$ or $b \leq a$ and if both are true, then $a = b$.

Instead of carrying out operations on the original objects we were counting, we can carry out symbolic operations on numbers: arithmetic.

$$1, 2, 3, \ldots, \infty?$$

Once the idea of number was invented, independent of fingers, it became clear that there was always a next number.

This was convenient (you'd never run out), but also, philosophically, a bit puzzling.

If numbers count *everything*, how do you count numbers?

The simplest idea is invent one more number, symbolized by ∞, to "count" the number of counting numbers.

Can Infinity Be A Number?

Although ∞ solves one problem (how many numbers are there?) if we try to treat it as another number, it creates new problems:

- Is $\infty + 1$ the number after ∞? Is that a different number or the same?
- If $\infty + 1 = \infty$, then $1 = 0$?
- What is $1/\infty$?

So maybe the question to ask isn't "Does infinity mean everything?" but "Does infinity mean anything?"

Ideas About Infinity

1. You can't get something from nothing.
2. You can never actually do an infinite number of things.
3. The whole must be greater than any of its parts.
4. Adding something to a quantity makes it larger.
5. Infinity is "how many" counting numbers there are;
6. Infinity is like "the last counting number";
7. Any two infinite things are the same size.
8. An infinite number of zeros is still zero;
9. A line one inch long is made up of an infinite number of points of zero length;
10. There are more points inside a square than on a line segment.

So it seems that, before we can understand what infinity is, we need a better definition of it...but this seems to be like a cat chasing its tail!

"If any philosopher had been asked for a definition of infinity, he might have produced some unintelligible rigmarole, but he would certainly not have been able to give a definition that had any meaning at all."
Bertrand Russell, mathematician (and philosopher).

A Mathematical Quest for Definition

Infinity seems to be an important concept, and yet our intuitive ideas seem to result in nonsense; in particular, the rules we have worked out for finite quantities don't seem to apply.

This is a perfect case for mathematical investigation which will:

- seek a simple, precise definition corresponding to intuition;
- determine what sorts of operations can involve infinity;
- make a unified system of finite and infinite numbers;
- decide what questions are beyond the scope of the definition;
- investigate the implications of the definition.
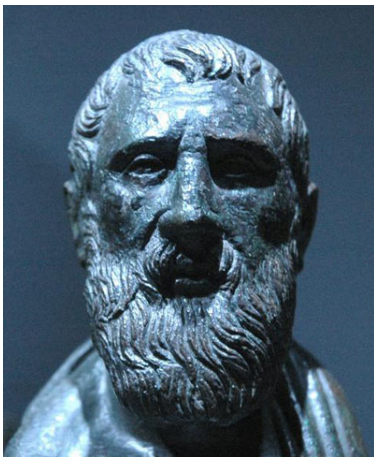
1. As Easy as 1, 2, 3?
2. **Paradoxes of Infinity**
3. Euclid's Prime Proof
4. Cantor Defines Infinity
5. Transfinite Numbers
6. How Many Points in a Line?
7. The Stairway to Heaven
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Zeno's Paradoxes

Paradox: something that contradicts your belief or knowledge.

The oldest paradoxes about infinity are associated with a Greek philosopher named Zeno of Elea.

Zeno's Paradoxes

Zeno pointed out that we believe that we can slice a piece of cheese into two pieces which we can put back together to get the original piece. But then we can also slice into 4, or 8, or as many pieces as we like. There seems no reason that the cheese could not be cut into an infinite number of equal pieces.

Then how big is each piece?

If the pieces have a nonzero size, then since there are infinitely many of them, the cheese itself must have an infinite size!

Therefore the pieces must have zero size. But doesn't that mean the cheese itself has zero size?

Zeno conclude that there is only one thing in the universe, and it cannot be divided at all!

Zeno's Paradoxes

Zeno also considered the paradox of *Achilles and the Tortoise*.

Suppose Achilles races a tortoise which is given a small head start. Then Achilles cannot win! By the time Achilles reaches the tortoise's initial position, the tortoise has moved ahead. When he reaches *that* position, again the tortoise has moved ahead. Every time the tortoise manages to move ahead. So the tortoise is never passed, and must win.

Zeno said that this proves that motion is an illusion!

Zeno's Paradoxes

If you think about this argument for a minute, it's not even important that the tortoise moves. Before catching up, Achilles has to reach half the distance, but then he still has another half to go. If he goes half of that distance (1/4 of the total), he still has 1/4 left. Each time he runs half of the remaining distance, he has another equal distance left to cover! According to this argument, Achilles couldn't even beat a **stick that had a head start!**

And what's worse, you can argue that Achilles can never even start the race. Before he can run the whole distance, he has to run half. But before he can run half, he has to run a quarter, and so on. In other words, there is no distance that Achilles can cover without covering another (smaller) distance first. But if every task requires you to complete another one first, you can't do any of them, right?

A Peculiar Example of Infinite Arithmetic

For example, in ancient times, there were many who believed the universe was eternal, and others who believed that it had come into being at a definite time. Without evidence, it would seem either case could be true.

But the philosopher John Philiponous of Alexandria scornfully denounced his opponents, and declared that, *as a matter of logic*, the age of the universe must be a finite number.

For let **Y** represent the age of the universe in years. Suppose that **Y** is infinite. Now we could also measure the age of the universe in months. This would be the number 12***Y**. But 12***Y** is obviously much bigger than **Y**, and yet *nothing* is bigger than infinity.

Therefore **Y** must be a finite number!

Albert of Saxony's Wooden Beam

Medieval philosophers argued whether infinity was logically consistent. Suppose an infinite object existed, for instance. Would this imply something logically impossible?

The philosopher Albert of Saxony considered whether it was possible to have a beam of wood that is infinitely long. Let us suppose only one end is infinite, and that the beam of wood has a 1 foot square cross section.

Then we can saw off a 1x1x1 foot cube of wood, Then we can saw off 8 more such cubes, surrounding the original cube to make a 3x3x3 cube of wood. Then we can saw off more cubes to make a 5x5x5 cube of wood. It follows that we can use the infinite beam to make a cube as big as we want, and thus, to fill the entire universe.

Infinity may not be logically impossible, but this would be one consequence.

Here the first 27 blocks of the beam have been arranged to begin filling up all of space.

Aristotle Restricts The Idea of Infinity

The philosopher Aristotle could not find any reasonable way to speak of infinity. But he recognized that it was necessary to have some concept that allowed things like counting to go on.

But since every counting operation eventually stopped, he decided to make a distinction. An operation like counting was **potentially infinite** because, although it would stop at some finite time, we could never place a limit on the operation beforehand. Philosophy and mathematics could accept potentially infinite operations and objects.

But the **actual infinite** was a contradictory and nonsensical idea. There are no infinite objects, and infinite operations are not possible. Mathematicians should never use the word infinity. In this way, all the paradoxes could be avoided.

"But my argument does not anyhow rob mathematicians of their study, although it denies the existence of the infinite in the sense of actual existence as something increased to such an extent that it cannot be gone through; for, as it is, they do not need the infinite or use it, but only require that the finite straight line shall be as long as they please. Hence it will make no difference to them for the purpose of proofs."
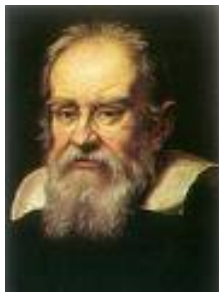Aristotle.

Galileo isn't Afraid of Infinity

1	2	3	4	5	6...
⇕	⇕	⇕	⇕	⇕	⇕ ..
1	4	9	16	25	36...

Galileo matched a list of numbers to a list of their squares. Both lists can be continued indefinitely, but one is a subset of the other! Many people (not Galileo!) argued that this was nonsense, since the whole must be greater than any of its parts.

Galileo isn't Afraid of Infinity

> *"So far as I see, we can only infer that the totality of all numbers is infinite, that the number of squares is infinite, and that the number of their roots is infinite; neither is the number of squares less than the totality of all numbers, nor the latter greater than the former; and finally, the attributes 'equal', 'greater' and 'less' are not applicable to infinite, but only to finite, quanitities."*
> Galileo

Galileo says it's obvious that infinite things exist, and if they don't behave the way we expect, then maybe that's because of our limited (finite) experience.

We will return soon to this very modern outlook!

Capturing Infinity

1. As Easy as 1, 2, 3?
2. Paradoxes of Infinity
3. **Euclid's Prime Proof**
4. Cantor Defines Infinity
5. Transfinite Numbers
6. How Many Points in a Line?
7. The Stairway to Heaven
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This is the oldest surviving fragment of Euclid's geometry book, the most famous mathematics book in history. This copy was made in 100 AD.
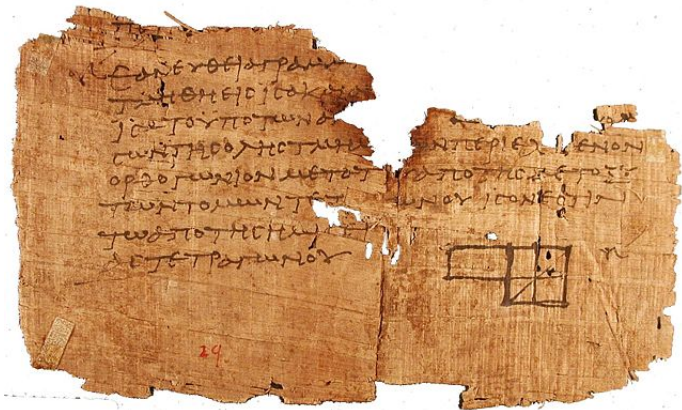
Euclid's Prime Proof

The ancient Greeks knew about prime numbers, that is, numbers like 2, 3, 5, 7, 11, 13, 17 which can't be divided by any other number except 1. Euclid proved a statement about them:

> *"Prime numbers are more than any assigned multitude of prime numbers."*
> Euclid

Because of Aristotle's objections, Euclid did **not** say:

> The list of all prime numbers is infinite.

Why complain about Aristotle's rule? Well, even though the two statements are equivalent, isn't it much more natural to think in terms of the second one? Even if we are forbidden to say infinity, it seems like we still think that way!

Euclid's proof is worth describing; it's simple, it's indirect, and it's a proof by contradiction. And we will need this same kind of proof when we are ready to talk about infinity.

Euler's task looks very difficult. It is a statement that says that **every** possible finite list of primes is incomplete. I know how to prove that for one particular list, but how do I do it for all possible lists?

Euclid took the roundabout approach, by considering the possible truth of the opposite of his statement:

It is possible to make a complete (and finite) list of prime numbers.

OK, said Euclid, suppose there is such a list: $P_1, P_2, ..., P_N$.

We agree every number is either a prime or is divisible by a prime (and hence is called a *"composite number"*).

I will now show that if you can make a finite list of all primes, I can define a number Q which is going to give us big problems.

Q is computed by multiplying together all the primes on the list, and then adding 1.

$$Q = P_1 * P_2 * ... * P_N + 1$$

Now the number Q is either a composite number or a prime.

Notice that when we divide Q by any prime number P_i on our complete list, we get a remainder of 1. Since the list supposedly contains all prime numbers, this means that Q cannot be a composite number.

That means Q must be prime. But we can easily see that Q is bigger than every prime number on our list of all prime numbers.

So if a finite list of primes is possible, then it must also be possible to have a number that is not composite and not prime!

So a finite list of prime numbers is **not possible**.

So every finite list of primes is incomplete.

Notice, again, that Euclid has just proved that there are an infinite number of primes, without ever using that word or that idea.

Moreover, he started out having to prove a statement about every possible finite list of primes, that is, he had to prove an infinite number of things.

His proof avoided this infinite task; instead, he supposed he was wrong, and that there was a single list of all primes, and he showed this could not happen.

Greek mathematics did not understand infinite objects; Euclid was careful to drive on the finite side of the road.

Here is an English translation of Euclid's original proof:

Prime numbers are more than any assigned multitude of prime numbers.

Let A, B, and C be the assigned prime numbers.

I say that there are more prime numbers than A, B, and C.

Take the least number DE measured by A, B, and C. Add the unit EF to DE to form DF.

Then DF is either prime or it is not.

First, let DF be prime. Then the prime numbers A, B, C, and DF have been found, which are more than A, B, and C.

Euclid's Prime Proof: The Real Version

Next, let DF not be prime. Therefore, it is measured by some prime number. Let it be measured by the prime number G.

I say that G is not the same with any of the numbers A, B, and C. If possible, let it be so.

Now A, B, and C measure DE, therefore G also measures DE. But G also measures DF. Therefore G, being a number, measures the remainder, the unit EF, which is absurd.

Therefore, G is not the same with any of the numbers A, B, and C. And by hypothesis, G is prime. Therefore the prime numbers A, B, C, and G have been found which are more than the assigned multitude of A, B, and C.

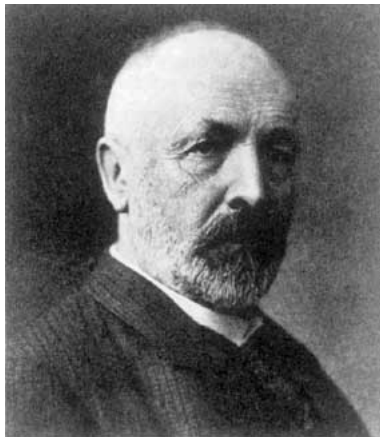Therefore, prime numbers are more than any assigned multitude of prime numbers.

Georg Cantor: 1845-1918

Stumbling Into Infinity

Georg Cantor was working on problems involving functions defined by Fourier series. In particular, he was looking for solutions to equations of the form **f(x)=0**.

We are familiar with this problem from high school algebra. A typical linear equation, such as $y = 3x - 15$, will (almost always) have exactly one solution, in this case when $x = 5$.

The slightly more complicated quadratic equation, such as $y = x^2 - 2x - 3$, can have two solutions, $x = -1$ or $x = +3$, but it's also possible to have just one solution, or none.

Using the technique of graphing, and realizing that every quadratic equation will graph as a parabola, we can see why a quadratic equation has 2, 1 or 0 solutions: a parabola can cross the x-axis at 2, 1 or 0 distinct values of x.

Quadratic equations with 0, 1 and 2 solutions
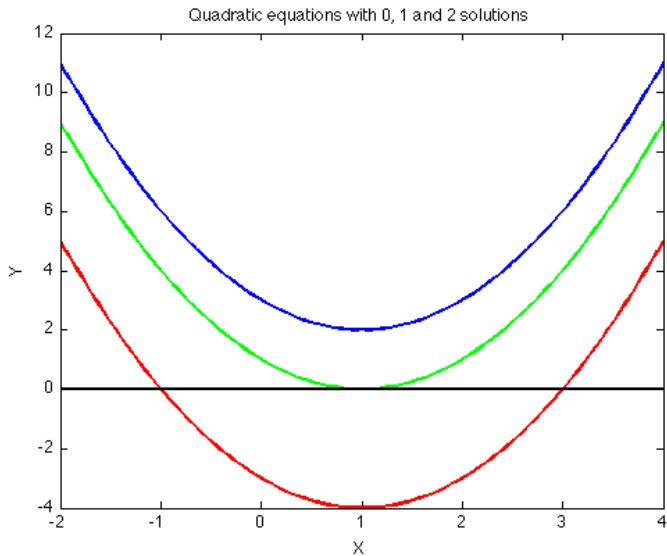
Cantor was working with functions defined by *Fourier series*. The important thing to note is that such functions are combinations of sine curves, which are very wiggly. Each time such a function crosses the x axis represents a solution of the equation $f(x) = 0$, and this means it's easy for such a function to have many zeroes, perhaps...an infinity of them.

Even in his time, the issue of infinity was one that made mathematicians uncomfortable. To say that a function had an infinite number of solutions was considered a careless statement, since infinity did not have any real meaning. It was preferable to say that the number of solutions was *"unbounded"*, that is, there was no finite number that could describe it.

y = - sin(x) sin(7x) + 1/2
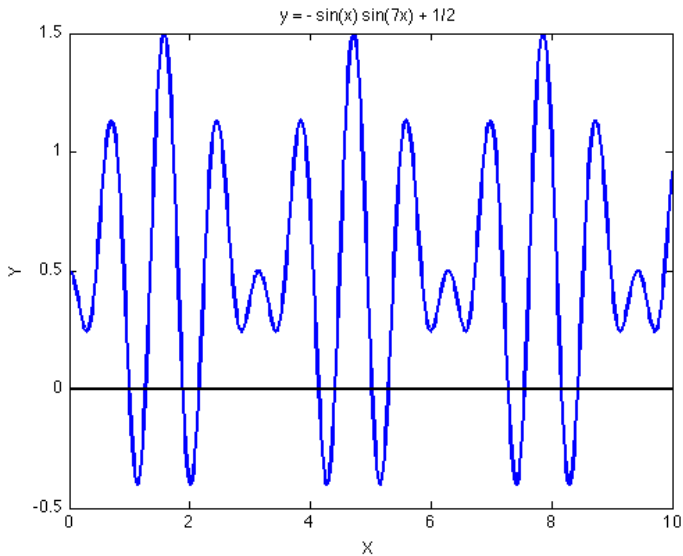
During his work, Cantor began to feel that it was unnatural to pretend that infinite sets didn't exist, or were inherently paradoxical. He began to believe it possible to construct a mathematically rigorous theory of infinite objects.

The tool he developed was **Set Theory**, the study of collections of abstract objects.

Cantor decided to make a very general definition of sets:

Intuitive Notion of a Set

"A set is any group of objects that can be thought of as a whole."
Georg Cantor

A set could contain numbers, letters, names, or even more sets.

If the set was small, it could be described as a list:

$$A = \{1, 2, 4, 97\}$$

The *empty set*, containing nothing, was symbolized by $\emptyset$:

$$\emptyset = \{\ \}$$

A set could also be described by a property its elements shared:

$$B = \{d \mid d \text{ is a divisor of } 36\} = \{1, 2, 3, 4, 6, 9, 12, 18, 36\}$$

But if we try to "define" the natural numbers as a set, it's not really clear what we mean and whether this is a legitimate definition:

$$\mathbb{N} = \{1, 2, 3, ...\}$$

For that reason, we will prefer to make an explicit axiom that we can generate an infinite set:

Axiom of Infinity

There is a set S which contains the empty set, and such that, if e is an element of S, then $\{e, \{e\}\}$ is also an element of S.

Each natural number n can be represented by the set of all natural numbers that are less than it. We represent 0 by $\{\}$. We represent 1 by $\{0\} = \{\{\}\}$. We represent 2 by $\{0, 1\}$ and so on. Now we recognize that the set S can be regarded as a form of $\mathbb{N}$.

$$\mathbb{N} = \{1, 2, 3, ...\}$$

Now Cantor has explicitly said that he is considering the set of natural numbers *as a whole*, that is, he is considering that the list above has been completed. In a sense, by adding the closing curly bracket to the description of $\mathbb{N}$, Cantor was declaring that he was not afraid to face a completed infinity.

Then Cantor began to consider all the solutions to an equation as **the set of solutions**. Such a set could be finite, infinite, or even "empty" (if there were no solutions to an equation).

Stumbling Into Infinity

Given sets **A** and **B**, he could define the *intersection* $A \cap B$ containing their common elements, and the *union*, $A \cup B$, containing all the elements in one or the other.

If every element of **A** was also in **B**, then **A** was a subset of **B**, or $A \subseteq B$. If **B** included all the elements of **A**, but also other elements that were not in **A**, then **A** was said to be a strict subset of **B**, or $A \subset B$.

Axiom of Equality

Two sets A and B (finite or infinite) are **equal** if and only if $A \subseteq B$ and $B \subseteq A$.

Stumbling Into Infinity

The number of elements in a set A was called its *cardinality*, symbolized by $\#A$.

Since $\subset$, for comparing sets, is similar to $<$, for comparing numbers, Cantor expected the following:

(INCORRECT) Relation

Given two sets A and B, then:
$A \subseteq B \Leftrightarrow \#A \leq \#B$ and
$A \subset B \Leftrightarrow \#A < \#B$.

These statements are certainly true for finite sets. But once again, infinity refused to behave.

The problem arises when we compare, for instance, the natural numbers $\mathbb{N}$ and the even numbers $\mathbb{E}$. Since $\mathbb{E} \subset \mathbb{N}$, we must have $\#\mathbb{E} \leq \#\mathbb{N}$ But if we expect $\#\mathbb{E} < \#\mathbb{N}$, then suddenly we have an infinite set that is smaller than the smallest infinite set! of these numbers, then indeed we have $\mathbb{S} \subset \mathbb{N}$, the first set is strictly a subset. Therefore, if we claim that $\#\mathbb{N} = \infty$, then

(CORRECT) Relation

Given two sets A and B, then:
$A \subset B \Rightarrow \#A \leq \#B$ and:
$A \subset B \Leftarrow \#A < \#B$.

This is fine as long as one set is a subset of another, but two sets need not have any elements in common. Can we still compare their sizes?

Stumbling Into Infinity

Axiom of Comparability

Any two sets A and B (finite or infinite) have **comparable set size**:

either $\#A \leq \#B$ or $\#B \leq \#A$ and if both are true, then $\#A = \#B$.

$\#A \leq \#B$ if and only if there is a mapping from A to B which maps each element of A to a distinct element of B.

Hence, if it can be shown that there is no such mapping from A to B, then by the axiom of comparability, it must be the case that $\#B < \#A$.

As he developed his ideas of set theory, he realized that many things that "worked" for finite sets were logically the same for infinite sets, and he began to think that there were are least ways in which infinity could be allowed in mathematics. But then he began to wonder about the problem that had occurred to Galileo, namely, the matching problem. Could one infinite set be just as large as another but at the same time be smaller?

After a long struggle, Cantor realized that this problem could be resolved once we realized that infinite sets were different from finite sets in an important way.

Cantor Defines Infinity

> *"An infinite set is precisely any set that can be placed in 1-to-1 correspondence with a proper subset of itself."*
> Georg Cantor

With this statement, Cantor declared that many of the paradoxes about infinity were, in fact, simply mistaken attempts to describe infinity using ideas about finite sets.

In this way, Galileo's paradox is actually a demonstration that the natural numbers $\mathbb{N}$ are an infinite set.

We'll shorten "a 1-to-1 correspondence" to simply *a matching*. We symbolize a matching between sets **A** and **B** by

$$A \sim B$$

The Definition of an Infinite Set

It's important that we agree that this definition "captures" infinity, that is, that what we are defining here has a close relationship with the intuitive concept of infinity.

The definition guarantees that an infinite set will remain infinite if:

- you add one item;
- you add an infinite number of items;
- you subtract one item;
- you **carefully** subtract an infinite number of items.

Cantor's Discovery

So to compare two sets, we don't have to compare the numbers that represent their sizes. (After all, we're not sure we can do arithmetic with ∞). Instead, we simply ask whether there is a matching or not.

> *"Two sets have the same cardinality precisely if they can be matched."*
> Georg Cantor

> *"If A matches a subset of B, but B does not match a subset of A, then $\#A < \#B$ "*
> Georg Cantor

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Cantor decided not to use the symbol ∞ to denote the "size" (cardinal number) of the counting numbers. Instead, he picked the Hebrew letter $\aleph$ ("aleph").

For reasons to become clear later, he gave it the subscript 0.

$$\mathbb{N} = \{1, 2, 3, ...\}$$
$$\#\mathbb{N} = \aleph_0.$$

$\aleph_0$ was Cantor's first example of a **transfinite cardinal number**.

Such numbers measure the size of infinite sets.

Some count this way: 1, 2, 3, ... and others count 0, 1, 2, 3, ...

Call these sets $\mathbb{N}^+$ and $\mathbb{N}_0$.

Is it true that $\mathbb{N}^+ \sim \mathbb{N}_0$?

Certainly! Here is a matching:

1	2	3	4	5	6...
$\Updownarrow$	$\Updownarrow$	$\Updownarrow$	$\Updownarrow$	$\Updownarrow$	$\Updownarrow$ ..
0	1	2	3	4	5...

This is puzzling result #1 in transfinite arithmetic:

$$1 + \aleph_0 = \aleph_0$$

Transfinite Numbers: $1 = 0$?

So why doesn't the equation

$$1 + \aleph_0 = \aleph_0$$

imply

$$1 = 0?$$

In Cantor's arithmetic, you can't directly add or subtract cardinals.

You first find sets A and B so that $A \sim B$
then you can write $\#A = \#B$!

To "prove" that $1 = 0$, we'd have to match a set of 0 elements
and a set of 1 element.

The set of integers is $\mathbb{Z} =$ ..., -3, -2, -1, 0, 1, 2, 3, ...
Since $\mathbb{N} \subset \mathbb{Z}$, we know $\#\mathbb{N} \leq \#\mathbb{Z}$.
But is a matching possible?

$...-3$	-2	-1	0	1	2	$3...$
$\Updownarrow$	$\Updownarrow$	$\Updownarrow$	$\Updownarrow$	$\Updownarrow$	$\Updownarrow$	$\Updownarrow$ ..
$...7$	5	3	1	2	4	$6...$

Therefore, $\mathbb{N} \sim \mathbb{Z}$ and so $\#\mathbb{N} = \#\mathbb{Z}$ or:

$$\aleph_0 = \aleph_0 + 1 + \aleph_0 = \aleph_0 + (1 + \aleph_0) = \aleph_0 + \aleph_0 = 2 \times \aleph_0$$

So puzzling result $\#2$ in transfinite arithmetic is:

$$2 \times \aleph_0 = \aleph_0$$

The set of all fractions or "rational numbers" is denoted by $\mathbb{Q}$. Surely this is bigger! To simplify things, let's just consider strictly positive fractions, which we can write as $\mathbb{Q}^+$.

The positive numbers $\mathbb{N}^+$ are a subset of $\mathbb{Q}^+$. They are the fractions with 1 as a denominator. But there are "just as many" fractions with 2 as a denominator, and another set with 3, 4, ... In fact, $\mathbb{Q}^+$ can be thought of as *infinitely many* copies of the counting numbers.

It might seem that there is no way to match $\mathbb{N}^+$ to $\mathbb{Q}^+$.

Let us imagine the fractions laid out in a table. In the first row are the fractions with 1 as a denominator, and in the second those with 2 and so on.

Our task of "counting" these fractions can be thought of as requiring us to imagine writing all the numbers in this table, in such a way that every fraction is "scheduled" to be written at a definite, finite time.

So if we simply plan to write the entire first row, *then* the second row, we'll never get to the second row, because it will take *forever* to finish the first row. So that idea fails.

So we must start the second row before we have completely finished the first one. We could write one number from the first row, and one from the second, then go back to the first row, and alternate. That would guarantee we got two rows taken care of...but we'd never get to the third row.

But what if we try zigzagging in a creative way? Start the first row, then start the second, and go back to the first, then start the third row, go back to the second row, then back to the third, and so on.

So now we know that $\mathbb{Q}^+ \sim \mathbb{N}^+$, and therefore $\#\mathbb{Q}^+ = \#\mathbb{N}^+$

Now the elements of the set $\mathbb{Q}^+$ are pairs of elements of $\mathbb{N}^+$

In this case, we say that $\mathbb{Q}^+$ is the *product* of $\mathbb{N}^+$ with itself; this is symbolized by $\mathbb{Q}^+ = \mathbb{N}^+ \otimes \mathbb{N}^+$

The number of elements in a product is simply the product of the number of elements in each factor, so $\#\mathbb{Q}^+ = \#\mathbb{N}^+ \times \#\mathbb{N}^+$

So puzzling result #3 in transfinite arithmetic is:

$$\aleph_0 \times \aleph_0 = \aleph_0$$

It's beginning to seem like all infinities are the same size!

How Many of These Things are Denumerable?

If a set of things is infinite, and can be matched with the counting numbers, we say it is *countably infinite* or *denumerable*.
Can you show that each of these sets is denumerable?

- all possible English sentences.
- all possible books in English.
- all possible musical pieces.
- all possible images on a computer screen.
- all possible movies.
- the number of times you can write an "X" of any size on one sheet of paper without crossing any others (this is **not** true for "O"!)

How Many Points in a Line?

The counting numbers are pretty obviously infinite.

Now let's try to find a different type of infinity, by drawing a line just one inch in length, which is technically a *line segment*.

What is this line made of? Euclid will tell us that the line consists of "points" and that these points have no length or width.

And now we ask the question:

Exactly how many points are there in a line?

How Many Points in a Line?

Each point can be labeled by its distance from the left end, so points match numbers from 0.0 to 1.0.

Consider the following geometrical statement:

Between two distinct points on a line is another distinct point

Arithmetically, if our points are **a** and **b**, one point that is between them is $\frac{a+b}{2}$.

Based on that reasonable statement, we can conclude:

The number of points in the line segment must be infinite.

And now a real question arises:

Are there more points in a line than there are counting numbers?

Up to this point, every infinite set has seemed to be the same size as the counting numbers. Cantor had shown this for the set of all algebraic numbers (essentially, square roots, cube roots, and various combinations).

The remaining case was $\mathbb{R}$, the set of all real numbers.

It would be enough to count just the real numbers between 0 and 1, symbolized by $\mathbb{R}[0, 1]$. If they could be counted, so could $\mathbb{R}$.

So a solution to the problem could be thought of as **a numbered list** of all the elements in $\mathbb{R}[0, 1]$, our 1 inch line.

So here's one idea that almost seems to work. Turn the counting numbers around, with a decimal point in front.

$\mathbb{N}$	$\mathbb{R}[0, 1]$
0	.0
1	.1
2	.2
3	.3
4	.4
...	...
128	.821
129	.921
130	.031
...	...
12345	.54321
...	...

Unfortunately, this list only includes some real numbers. Actually, it only includes rational numbers. And it doesn't even include all of them. It only includes rational numbers with a denominator that is 10, 100, 1000 and so on.

Ask yourself where, on this list, does the number $\frac{1}{3}$ occur?
It never shows up!

Where on this list does the nonterminating decimal $\frac{pi}{4}$ occur?
Nowhere!

So this brilliant idea actual is a dead end.

After many tries at making a list, Cantor thought perhaps he was wrong. Perhaps you couldn't list the real numbers because the real numbers were "more infinite" than the natural numbers, in a way that the integers and rationals and algebraic numbers were not.

So Cantor tried the approach that Euclid did. Instead of looking for a list, he supposed that someone else had made such a list. Now granted, the list was infinite, but every real number must occur at a finite position on the list, in the same way that every counting number shows up finitely soon, even though there are an infinite number of them.

Cantor proceeded to prove that there was at least one number missing from this "complete" list.

Real Numbers: Suppose We Had A List

Keep in mind that each number on this list has an infinite number of digits, so the list is infinite *across* as well as down!

$\mathbb{N}$	$\mathbb{R}[0, 1]$
1	0.1234567890...
2	0.3924567833...
3	0.7134567395...
4	0.3957382988...
5	0.1237685907...
6	0.9837485922...
7	0.1345673974...
8	0.8395738295...
9	0.0237685913...
10	0.2983748591...
...	...

Real Numbers: Scan the List Diagonally

Cantor considered the number formed by the first digit of the first number, the second digit of the second number, and so on:

$\mathbb{N}$	$\mathbb{R}[0, 1]$
1	0.1234567890...
2	0.3924567833...
3	0.7134567395...
4	0.3957382988...
5	0.1237685907...
6	0.9837485922...
7	0.1345673974...
8	0.8395738295...
9	0.0237685913...
10	0.2983748591...
...	...

The diagonal number is .1937683211....

Cantor then increased each digit of .1937683211... by 1, (but 9's become 0's), to get the modified number .2048794322....

He pointed out this number cannot be on the list.

It can't, because

- its first digit differs from that of the number in position 1,
- its second digit differs from that of the number in position 2,
- its **n**-th digit differs from that of the number in position **n**,
- and so on.

So this "complete" list of real numbers is actually incomplete.

You might imagine you could fix things by adding this missing number. But the same argument will find you *yet another* missing number.

This argument shows it is impossible to count the real numbers. You cannot find a 1-to-1 matching between $\mathbb{N}$ and $\mathbb{R}$.

Cantor symbolized the cardinality or size of $\mathbb{R}$ by c.

With his "diagonal argument", he had shown that $\aleph_0 < c$.

He wrote to a friend in some astonishment:

"I see it, but I can't believe it!"
Georg Cantor

With the discovery that $\aleph_0 < c$, Cantor realized that the statement "all infinities are the same size" was false. There is more than one size of infinite set.

$\aleph_0$ was certainly the first step into infinity (can you prove there is no infinite set strictly smaller than $\mathbb{N}$?), and c was a further step.

Was c the very "next" step after $\aleph_0$, (so that we could rename c to $\aleph_1$) or were there intermediate steps?

And were there infinite sets that were strictly larger? If so, was there a way to create them?

Since $\mathbb{R}$ counts the points in a line, perhaps a bigger infinity is the number of points in a plane, symbolized by $\mathbb{R} \times \mathbb{R}$.

But Cantor found a matching between points in a line and a plane.

Given a point in the unit square such as $(x, y) = (.012345, .56789)$, we can construct its matching point in the unit line by alternating the digits of x and y to get .0516273849.

With some adjustments, you can match all points in the plane with the points in a line.

So $c \times c = c$.

The same trick means that $c \times c \times c = c$.

So c counts the points in a line, in a plane, and in all space.

Since c counts the points in a 1 inch line segment as well, all these objects have the same number of points.
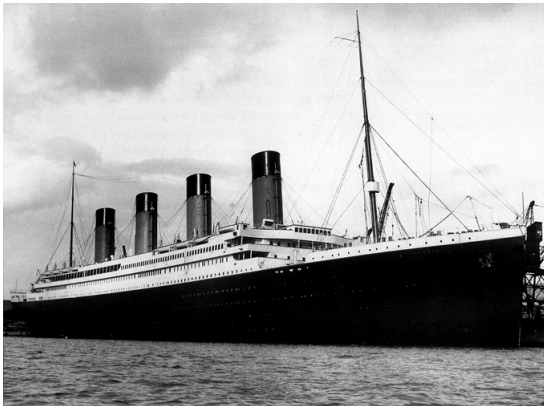
Remember Albert of Saxony's paradox, where an infinite piece of wood can be used to fill space?

Cantor has shown something more shocking. The points on a 1 inch line segment (which has no area, and no volume) can be rearranged to fill an entire line, an entire plane, or all of space!

It's stranger than saying you could build the Titanic from an inch of wire!

Cantor had achieved his goal of finding a set that was bigger than the counting numbers $\mathbb{N}$. Naturally, he wondered whether there were even larger sets.

Since the plane $\mathbb{R}^2$ and space $\mathbb{R}^3$ were no bigger than $\mathbb{R}$, Cantor tried to think of other mathematical operations or objects that might reach a higher level.

Cantor's breakthrough came when he tried to explain why c had to be bigger than $\aleph_0$.

Capturing Infinity

Different Sizes of Infinity?

One way to look at a real number is as a sequence of binary digits.

Every real number in [0,1] corresponds to a distinct set of binary digits.

$$\frac{\pi}{4} = 0.785398... = 0.110010010000111111...$$

If we write down the locations where a 1 occurs, we get a subset of the counting numbers.

$$\frac{\pi}{4} \rightarrow \{1, 2, 5, 8, 13, 14, 15, 16, 17, 18, ...\}$$

The correspondence between real numbers in [0,1] and distinct subsets of the counting numbers means that there are as many real numbers in [0,1] as there are elements of the power set of $\mathbb{N}$.

So $\mathbb{R} \sim \mathcal{P}(\mathbb{N})$

But the number of elements in the power set of a set **A** is $2^{\#A}$.

For this reason, the power set of **A**, which we write $\mathcal{P}(\mathcal{A})$, is also often written as 2^A.

In other words, an element of the power set of **A** (a subset of **A**) is defined by choosing 1 or 0, (yes or no), for each element of **A**.

Therefore, what Cantor realized was that the real numbers $\mathbb{R}$ could be matched with the power set of $\mathbb{N}$:

$$\#\mathbb{N} = \aleph_0 < 2^{\aleph_0} = \#\mathbb{R}$$

In other words, while addition and multiplication of cardinal numbers didn't produce a bigger infinity, exponentiation did. At least in this one case.

So Cantor wondered if this was always true. Given a set **A**, consider the power set 2^A. Is that set always strictly bigger?

Obviously, for a finite set **A** of size **n**, the power set is bigger, because $n < 2^n$.

But when the set **A** is infinite, we can't do arithmetic directly. We have to compare the actual sets.

To show that $\mathcal{P}(\mathcal{A})$ is strictly bigger than **A**, we have to compare the two sets, and show there is no way to match **A** to $\mathcal{P}(\mathcal{A})$.

To prove that this was impossible, Cantor used Euclid's idea of turning the argument around:

"Suppose there was such a matching...?"

Cantor supposes there was a matching between **A** and $\mathcal{P}(\mathcal{A})$.

Then we have a matching $a \leftrightarrow S_a$.

In some cases **a** is an element of S_a, and in some cases not.

Consider those **a** which are not elements of their match S_a.

Call **S*** the set of all such **a**. For some **a***, we have $a^* \leftrightarrow S^*$.

If **a*** is not an element of its matching subset **S***, then it qualifies for membership in **S***. If **a*** is an element of **S***, then it got there because it is not a element of its matching subset **S***.

Cantor had showed that, even when a set **A** is infinite, the power set must be strictly bigger.

If A is an infinite set, then 2^A is somehow a "more infinite" set.

Thus, starting with the infinite set $\mathbb{N}$, we can use exponentiation to get the strictly larger set $\mathbb{R}$. But now we can repeat this process, to consider the set $2^{\mathbb{R}}$, the set of all subsets of real numbers, which must even larger, and so on.

Cantor had shown that it is not true that "all infinities are the same size". Instead, he had shown that the two infinities we knew about were simply starting steps on an endlessly rising stairway.

So after our infinite sets $\mathbb{N}$ and $\mathbb{R}$, we can form another set $\mathbb{F}$ which is guaranteed to be bigger, namely $\mathbb{F} = 2^{\mathbb{R}}$, that is, the set of all subsets of $\mathbb{R}$.

Another way to describe this set is the set of all functions $f(x)$ on the real line taking values of 0 or 1: $\mathbb{F} = \{f | f : \mathbb{R} \to \{0, 1\}\}$.

The size of this set is 2^c and this is strictly bigger than c.

We could make an even larger set $\mathbb{G} = 2^{\mathbb{F}}$ and so on.

Capturing Infinity

The claim that c is the "next" infinity after $\aleph_0$ is known as Cantor's "Continuum Hypothesis".

Cantor spent his last years trying to prove it, but could not.

Two famous mathematicians, Kurt Gödel and Paul Cohen, were later able to show that the continuum hypothesis could never be proved from the basic laws of set theory, but that it also did not contradict those basic laws.

It is a little shocking to discover that some meaningful mathematical statements cannot be settled from our basic assumptions.

It is surprising to see that the pattern of mathematical ideas and arguments has recurred over many years.

Cantor's proof that the power set is strictly bigger uses ideas that Euclid used to show there can't be a biggest prime.

Galileo's idea of using matching to show that there were as many squares as counting numbers ended up being used by Cantor as his definition of an infinite set.

Cantor's diagonal argument, which he used to showed the real numbers cannot be counted, was reused by Alan Turing in a proof about computability, and by Kurt Gödel in his famous proof that mathematical systems are incomplete.

The fact that the Continuum Hypothesis cannot be proved from basic set theory recalls the fact that Euclid's parallel line postulate cannot be proved from the other axioms of geometry, although many people tried over the years.

Cantor gave infinity a logical base as a mathematical subject.

Cantor cleared up the paradoxes by realizing that the "laws" that infinity violated were laws that defined finite numbers.

He found the key to his theory when he defined infinite objects as exactly those that are "equal" to a proper subset of themselves.

"Infinite numbers, if they are to be thinkable in any form, must constitute quite a new kind of number, whose nature is an object of investigation, but not of our arbitrariness or prejudices."
Georg Cantor

Conclusion

The mathematician Hilbert judged Cantor's work as follows:

> *"The finest product of mathematical genius and one of the supreme achievements of purely intellectual human activity."*
> *David Hilbert*

He also made a more poetic statement that suggests his joy at having a logical way of viewing infinity:

> *"No one will expel us from this paradise Cantor has created for us."*
> *David Hilbert*