Formulas useful in astronomy

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Notation

A: the vernal equinox point on the celestial sphere

B: the northerly pole of the ecliptic

E, N, S, W: the eastern, northern, southern and western points on the horizon (which is the circle in which the horizontal plane through the observer cuts the celestial sphere).

O: the centre of the celestial sphere

P: the north celestial pole

Z: the observer's zenith

X^*: the point where the meridian (a semicircle from pole to pole) through a point X on the celestial sphere cuts the celestial equator.

α: right ascension

β: celestial latitude

δ: declination

ϵ: the obliquity of the ecliptic

ζ: azimuth, measured eastward from north

η: hour angle

θ: sidereal time

λ: celestial longitude

τ: altitude

ϕ: the observer's latitude

N.B. I use Greek letters for angles: ζ and τ are not standard notation. Throughout I neglect the fact that the earth is not a perfect sphere. For some purposes we can take the centre of the celestial sphere to be the observer instead of the centre of the earth. Only if we are dealing with the moon or dealing with the sun so precisely that its parallax matters will this make any difference.

Definitions (Illustrated in diagram 1)

Sidereal noon is the instant when A is at Z^* (due south)

Sidereal time is the time since then and is measured by the angle Z^*OA at 15° per hour.

The *hour angle* of X is the angle XPZ (= angle X^*OZ^*) measured eastward. Then

$$\theta = \eta + \alpha \tag{1}$$

Orthogonal coordinates

β and λ form a pair of orthogonal coordinates fixed in space; so do δ and α. We can transform between them as follows.

$\underline{\delta \text{ and } \alpha \text{ in terms of } \beta \text{ and } \lambda}$

$$\sin\delta = \sin\beta\cos\epsilon + \cos\beta\sin\lambda\sin\epsilon \qquad -90^\circ \leq \delta \leq 90^\circ \tag{2}$$

$$\tan\alpha = (\sin\lambda\cos\epsilon - \tan\beta\sin\epsilon)\sec\lambda \tag{3}$$

Choose the value of α that gives $\cos\alpha$ the same sign as $\cos\lambda$. If $\cos\lambda = 0$, α is 90° or 270°; a diagram easily shows which.

$\underline{\beta \text{ and } \lambda \text{ in terms of } \delta \text{ and } \alpha}$

$$\sin\beta = \sin\delta\cos\epsilon - \cos\delta\sin\alpha\sin\epsilon \qquad -90^\circ \leq \beta \leq 90^\circ \tag{4}$$

$$\tan\lambda = (\sin\alpha\cos\epsilon + \tan\delta\sin\epsilon)\sec\alpha \tag{5}$$

Choose the value of λ that gives $\cos\lambda$ the same sign as $\cos\alpha$. If $\cos\alpha = 0$, λ is 90° or 270°; a diagram easily shows which.
We have also

$$\cos\delta\cos\alpha = \cos\beta\cos\lambda \tag{6}$$

τ and ζ form a pair of orthogonal coordinates fixed with respect to the earth; so do δ and η. We can transform between them as follows.

<u>τ and ζ in terms of δ and η.</u>

$$\sin\tau = \sin\phi\sin\delta + \cos\phi\cos\delta\cos\eta \qquad -90^\circ \le \tau \le 90^\circ \tag{7}$$

$$\tan\zeta = \sin\eta/(\sin\phi\cos\eta - \cos\phi\tan\delta) \tag{8}$$

Choose the value of ζ that gives $\sin\zeta$ and $\sin\eta$ opposite signs

<u>δ and η in terms of τ and ζ.</u>

$$\sin\delta = \sin\phi\sin\tau + \cos\phi\cos\tau\cos\zeta \qquad -90^\circ \le \delta \le 90^\circ \tag{9}$$

$$\tan\eta = \sin\zeta/(\sin\phi\cos\zeta - \cos\phi\tan\tau) \tag{10}$$

Choose the value of η that gives $\sin\eta$ and $\sin\zeta$ opposite signs.

We have also

$$\cos\tau\sin\zeta = -\cos\delta\sin\eta \tag{11}$$

Sidereal time

At sidereal time θ:-

<u>The degree of the ecliptic on the horizon</u> is given by

$$-\tan\lambda = \cos\theta/(\sin\theta\cos\epsilon + \tan\phi\sin\epsilon) \tag{12}$$

One solution gives the degree that is rising, the other gives the degree that is setting.

<u>The degree of the ecliptic that is culminating</u> is given by

$$\tan\lambda = \tan\theta\sec\epsilon \tag{13}$$

The angle between the northward horizon and the upward ecliptic is

$$\arccos(\cos\epsilon\sin\phi - \sin\epsilon\cos\phi\sin\theta) \tag{14}$$

It is $90° + \phi + \epsilon$ at sunrise on the vernal equinox and at sunset on the autumn equinox (provided that the equinox occurs at sunrise or sunset respectively). This is the maximum value. The minimum is $90° + \phi - \epsilon$ at sunrise on the autumn equinox and at sunset on the vernal equinox.

Miscellaneous

The hour angle of a star on the horizon is given by

$$\cos\eta = -\tan\delta\tan\phi \tag{15}$$

and its azimuth by

$$\cos\zeta = \sin\delta\sec\phi \tag{16}$$

In each case, one solution gives the value at rising; the other gives the value at setting.

The longitude of the point on the ecliptic with right ascension α is given by

$$\tan\lambda = \tan\alpha\sec\epsilon \tag{17}$$

and its declination by

$$\tan\delta = \sin\alpha\tan\epsilon \tag{18}$$

N.B. This is the point on the ecliptic that culminates with any star whose right ascension is α; its longitude is the *polar longitude* of such a star

The altitude of the sun when the earth has rotated through an angle ψ since noon is given by

$$\sin\tau = \sin\phi\sin\delta + \cos\phi\cos\delta\cos\psi \tag{19}$$

The length d of daylight in hours when the sun has declination δ is given by

$$\cos(15d/2) = -\tan\phi\tan\delta \tag{20}$$

The rates of change $\dot{\delta}$ and $\dot{\alpha}$ of δ and α caused by precession at rate $\dot{\lambda}$ are given by

$$\dot{\delta} = \dot{\lambda} \cos\alpha \sin\epsilon \tag{21}$$

$$\dot{\alpha} = \dot{\lambda}(\cos\epsilon + \tan\delta \sin\alpha \sin\epsilon) \tag{22}$$

The changes in η and δ caused by refraction or parallax are found as follows. If τ changes to τ^*, causing η and δ to change to η^* and δ^*, then

$$\cot\eta^* = \cot\eta + \cos\phi \cos\tau csc\eta \sec\delta(\tan\tau^* - \tan\tau) \tag{23}$$

$$\sin\delta^* = \sin\delta \cos\tau^* \sec\tau + \sin\phi \cos\tau^*(\tan\tau^* - \tan\tau) \tag{24}$$

Location of a point with respect to a circle

If we know the angles subtended by arcs PQ and QR of a circle at the centre O and at a point E, we can locate E.

Let the radius of the circle be r. Let the angles subtended at E by PQ, QR and RP be α, β and γ, all measured in the same direction, so that $\alpha + \beta + \gamma = 360°$. Let $\lambda = POQ - \alpha$, $\mu = QOR - \beta$ and $\nu = ROP - \gamma$. Then

$$\frac{OE^2}{r^2} = \frac{\sin^2\lambda + \sin^2\mu + \sin^2\nu + 2\cos\alpha\sin\mu\sin\nu + 2\cos\beta\sin\nu\sin\lambda + 2\cos\gamma\sin\lambda\sin\mu}{\sin^2\alpha + \sin^2\beta + \sin^2\nu + 2\cos\lambda\sin\beta\sin\gamma + 2\cos\mu\sin\gamma\sin\alpha + 2\cos\nu\sin\alpha\sin\beta} \tag{25}$$

and

$$\tan QEO = \frac{\sin\alpha\sin\mu - \sin\beta\sin\lambda}{\cos\alpha\sin\mu + \cos\beta\sin\lambda + \sin\nu}. \tag{26}$$

Note: two points satisfy these equations, equally far from O in opposite directions. We need to calculate the angles subtended at each to pick the right one.

The azimuth of a star on the horizon is given by

$$\cos \zeta = \sin \delta \sin \phi \qquad (27)$$

One solution gives the azimuth at rising, the other at setting.

Orientation

If ξ is the angle between the line from the observer to the pole and a vertical plane through the observer with azimuth ζ, then

$$\sin \xi = \sin \zeta \sec \phi \tag{28}$$

Comments

If a building is being oriented north-to-south by using a vertical alignment thought to pass through the pole, (28) gives the error in orientation caused by an error in the position of the pole.

Hipparchus's Exegesis on Aratus gave, for many stars, the degree of the ecliptic that rises, sets or culminates when the star rises, sets or culminates. This cannot be observed directly but can be calculated from α and δ, coordinates used by Hipparchus.

The hour angle of the star when it rises or sets is given by (15); the hour angle is zero when it culminates. Then (1) gives the sidereal time. The degree of the ecliptic that rises or sets then is given by (12) and the degree that culminates by (13).

A fact useful in constructing an astrolabe is that the projection of a circle on the celestial sphere from a pole onto the plane of the equator is always a circle. I give a proof of this in the appendix after proofs of the formulas.

Formula	Proof on page
2 to 6	3
7 to 11	4
12	5
13 and 15 to 18	6
14	7
19	8
20	9
21 and 22	11
23 and 24	12
25 to 27	13 and 14
28	15

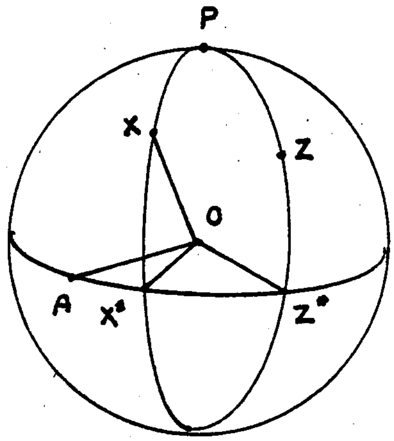


Diagram 1.

α : angle AOX^* measured eastward (counter-clockwise in this diagram)

δ : angle X^*OX

η : angle Z^*OX^*

θ : angle Z^*OA

ϕ : angle ZOZ^*

Relative to the earth the sky rotates westward, carrying with it the points X, A and X^*. The points P, Z and Z^* remain fixed. The angle Z^*OA increases at 360° per day.

Appendix

Let A, B, and C be the angles of a spherical triangle, and let a, b, and c be its sides. The following formulas, with proofs, are in any book of spherical trigonometry. The sine etc. of an arc is the sine etc. of the angle subtended at the centre of the sphere.

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C} \tag{I}$$

$$\cos a = \cos b.\cos c + \sin b.\sin c.\cos A \tag{II}$$

$$\cos A = -\cos B.\cos C + \sin B.\sin C.\cos a \tag{III}$$

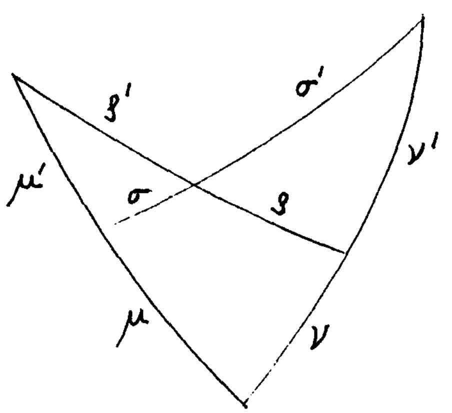
$$\tan B = \frac{\sin C}{\sin a.\cot b - \cos a.\cos C} \tag{IV}$$

$$\tan b = \frac{\sin a}{\cos a.\cos C + \cot B.\sin C} \tag{V}$$

If arcs of great circles are as in the diagram below, then

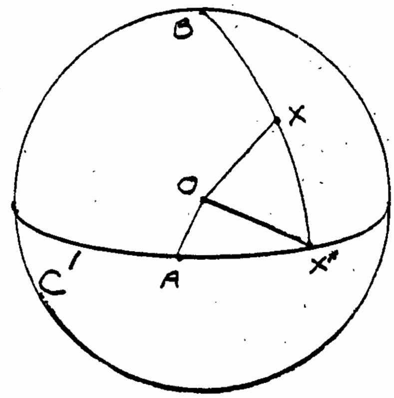
$$\frac{\sin\mu'}{\sin\mu} = \frac{\sin\rho'}{\sin\rho} \cdot \frac{\sin\nu'}{\sin(\nu + \nu')} \tag{VI}$$

$$\frac{\sin\mu}{\sin(\mu + \mu')} = \frac{\sin\rho}{\sin(\rho + \rho')} \cdot \frac{\sin(\sigma + \sigma')}{\sin\sigma'} \tag{VII}$$



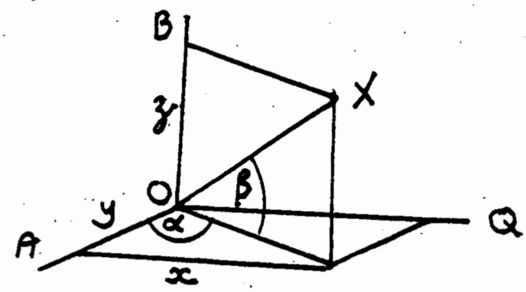
Orthogonal coordinates

Given a great circle C on the sphere, a point A on the circle from which to measure, a direction in which to measure, and a point B on the sphere, centre O, such that OB is perpendicular to the plane of C, orthogonal coordinates of a point X on the sphere are found as follows. Let the great circle from B via X first cut C in X*. The coordinate around is the angle AOX*. The coordinate across is the angle XOX*. It is positive if X is on the same side of the plane of C as B. Clearly $-90° \le XOX^* \le 90°$.



Transformation

Suppose that we have two systems of orthogonal coordinates on the sphere, whose basic circles intersect in A. Let the coordinates around and across be α, β for the first system and γ, δ for the second. Let D play the part of B for the second system. Let ε be the angle between the two great circles. let α and γ both be measured from A towards the points Q and R where the great circle from B via D first cuts the basic circles.



Introduce cartesian coordinates x, y, z with axes OQ, OA, OB for the first system, and coordinates u, v, w with axes OR, OA, OD for the second, and apply the transformation for a rotation of ε about the y-axis. Take the radius of the sphere to be 1.

$$x = \cos\beta.\sin\alpha \qquad u = \cos\delta.\sin\gamma \qquad u = x.\cos\varepsilon - z.\sin\varepsilon$$
$$y = \cos\beta.\cos\alpha \qquad v = \cos\delta.\cos\gamma \qquad v = y$$
$$z = \sin\beta \qquad w = \sin\delta \qquad w = x.\sin\varepsilon + z.\cos\varepsilon$$

Then

$$\left.\begin{aligned}
\cos\delta.\sin\gamma &= \cos\beta.\sin\alpha.\cos\varepsilon - \sin\beta.\sin\varepsilon \\
\cos\delta.\cos\gamma &= \cos\alpha.\cos\beta \\
\sin\delta &= \cos\beta.\sin\alpha.\sin\varepsilon + \sin\beta.\cos\varepsilon \\
\tan\gamma &= \frac{\sin\alpha.\cos\varepsilon - \tan\beta.\sin\varepsilon}{\cos\alpha}
\end{aligned}\right\} \qquad \text{(VIII)}$$

(If $\cos\alpha = 0$, X is on the great circle through B and D, and γ and δ are most easily found from a diagram.)

β and δ are both between $-90°$ and $90°$ (unless either is $\pm 90°$) and so have positive cosines. Then $\cos\alpha$ and $\cos\gamma$ have the same sign. Of the two values for γ choose the one that gives $\cos\gamma$ the same sign as $\cos\alpha$.

If either α or γ is measured in the opposite direction, change its sign. If either is not measured from A, subtract the appropriate constant.

Now apply these formulas to astronomy, using the notation listed earlier

To transform from ecliptical coordinates to equatorial, we use (VIII) with λ for α and α for γ.

$$\sin\delta = \sin\beta.\cos\varepsilon + \cos\beta.\sin\varepsilon.\sin\lambda \qquad -90° \leq \delta \leq 90°$$

$$\cos\delta.\cos\alpha = \cos\beta.\cos\lambda$$

$$\cos\delta.\sin\alpha = \cos\beta.\cos\varepsilon.\sin\lambda - \sin\beta.\sin\varepsilon$$

$$\tan\alpha = \frac{\cos\varepsilon.\sin\lambda - \tan\beta.\sin\varepsilon}{\cos\lambda} \qquad \text{if } \cos\lambda \neq 0$$

choose the value of α that gives $\cos\alpha$ the same sign as $\cos\lambda$.

To transform from equatorial coordinates to ecliptical interchange λ,β with α,δ and change the sign of ε.

$$\sin\beta = \sin\delta.\cos\varepsilon - \cos\delta.\sin\varepsilon.\sin\alpha \qquad -90° \leq \beta \leq 90°$$

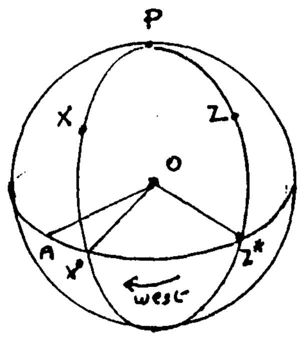
$$\cos\beta.\cos\lambda = \cos\delta.\cos\alpha$$

$$\cos\beta.\sin\lambda = \cos\delta.\cos\varepsilon.\sin\alpha + \sin\delta.\sin\varepsilon$$

$$\tan\lambda = \frac{\cos\varepsilon.\sin\alpha + \tan\delta.\sin\varepsilon}{\cos\alpha} \qquad \text{if } \cos\alpha \neq 0$$

choose the value of λ that gives $\cos\lambda$ the same sign as $\cos\alpha$.

Sidereal time



Sidereal noon is the instant when A is due south, i.e. at the point Z* where the meridian through Z cuts the equator.

Sidereal time θ is the time since then and is measured by the angle Z*OA at 15° per hour.

The *hour angle* η of X is the angle XPZ measured westwards.

If the meridian through X cuts the equator in X*, η = X*OZ*.

Because α is the angle X*OA and is measured eastwards,

$$\theta = \eta + \alpha.$$

η and δ are a pair of equatorial coordinates. We can transform between them and the horizontal system by using the triangle PZX.

Angle PZX = ζ. PZ = 90° - ϕ, PX = 90° - δ, ZX = 90° - τ, ZPX = $-\eta$ if X has not passed the meridian through Z; if it has, PZX = 360° - ζ and ZPX = η.

In either case, by (I), (III) and (V),

$$\sin\tau = \sin\phi.\sin\delta + \cos\phi.\cos\delta.\cos\eta \qquad -90° \leq \tau \leq 90° \qquad \text{(IX)}$$

$$\sin\delta = \sin\phi.\sin\tau + \cos\phi.\cos\tau.\cos\zeta \qquad -90° \leq \delta \leq 90°$$

$$-\cos\tau.\sin\zeta = \cos\delta.\sin\eta$$

$$\cos\eta = \frac{\sin\tau - \sin\phi.\sin\delta}{\cos\phi.\cos\delta}$$

$$\cos\zeta = \frac{\sin\delta - \sin\phi.\sin\tau}{\cos\phi.\cos\tau}$$

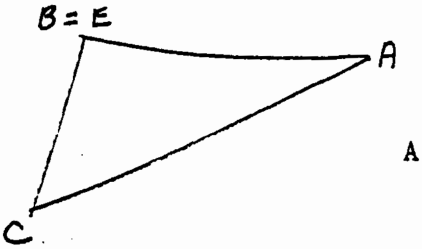
$$\tan\zeta = \frac{-\sin\eta}{\cos\phi.\tan\delta - \sin\phi.\cos\eta}$$

$$\tan\eta = \frac{-\sin\zeta}{\cos\phi.\tan\tau - \sin\phi.\cos\zeta}$$

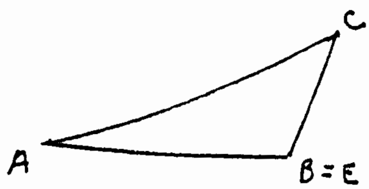
Choose the values of ζ and η that give $\sin\zeta$ and $\sin\eta$ opposite signs.

Time and the ecliptic

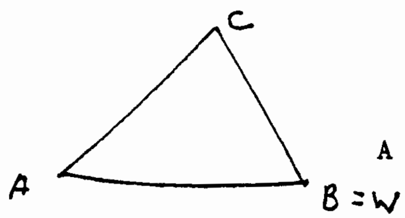
To find the degree of the ecliptic on the horizon at time θ, use the triangle ABC formed by the horizon, the equator and the ecliptic. B is either the east point E or the west point W of the horizon. Z^*A is θ, $Z^*W = 90°$, $Z^*E = 270°$. Four configurations are possible. In each, AB is the equator, BC the horizon, and CA the ecliptic.



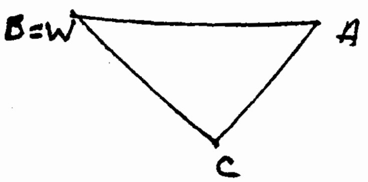
$A = \varepsilon$, $B = 90° + \phi$, $b = -\lambda$, $c = 270° - \theta$.



$A = \varepsilon$, $B = 90° + \phi$, $b = \lambda$, $c = \theta - 270°$



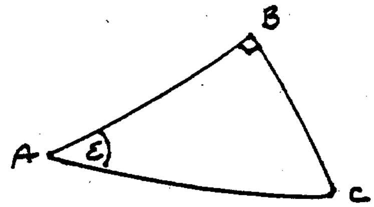
$A = \varepsilon$, $B = 90° - \phi$, $b = \lambda$, $c = \theta - 90°$



$A = \varepsilon$, $B = 90° - \phi$, $b = -\lambda$, $c = 90° - \theta$.

In all four cases (V) gives $\tan\lambda = \dfrac{-\cos\theta}{\tan\phi.\sin\varepsilon + \sin\theta.\cos\varepsilon}$.

A point on the (rotating) celestial sphere *culminates* when it is on the meridian through Z. It is then due south unless it is within $90° - \phi$ of the north pole.



The degree of the ecliptic that culminates at time θ is found from the triangle ABC formed by the equator AB, the ecliptic AC and the meridian BC through Z.

$A = \varepsilon$, $c = \theta$, $B = 90°$ and b is the degree λ sought.

From (V) $\tan\lambda = \dfrac{\tan\theta}{\cos\varepsilon}$

The degree of the ecliptic that culminates when a star culminates is sometimes called its *polar longitude*.

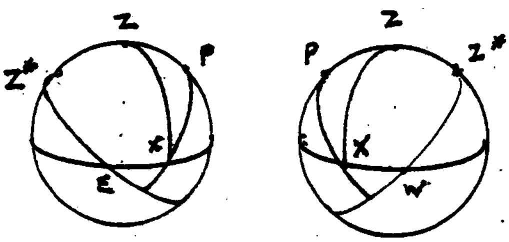
If C is the point on the ecliptic, then its declination, for which there is no technical term, is useful in following Ptolemy's treatment of precession. In the triangle ABC above, the declination of C is a and (V) yields

$$\tan a = \sin\alpha . \tan\varepsilon .$$

To find the hour angle of a star X with declination δ when it is on the horizon, set $\tau = 0$ in (IX):

$$\cos\eta = -\tan\delta . \tan\phi$$

One solution for η gives the value at rising, the other at setting.



We can also find the azimuth of the star from the triangle ZPX. $x = 90° - \phi$, $z = 90° - \delta$, $p = 90°$. If the star is rising it is on the eastern horizon, $Z = \zeta$.

If it is setting in the west, $Z = -\zeta$.

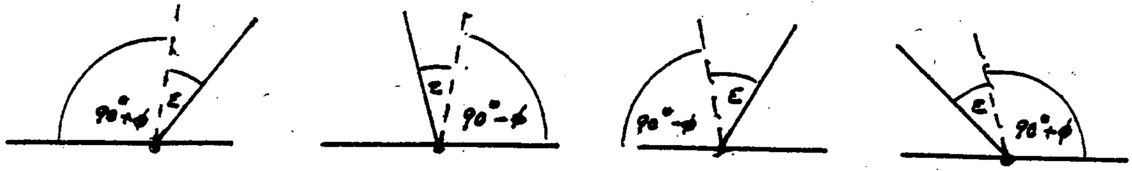
In either case, the formula $\cos z = \cos p . \cos x + \sin p . \sin x . \cos Z$ yields

$$\sin\delta = \cos\phi . \cos\zeta .$$

One solution for ζ gives the azimuth at rising, the other at setting.

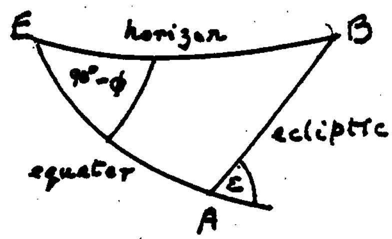
The angle between the horizon and the ecliptic.

At sunrise or sunset at the equinox the sun is at the east or west point of the horizon. The angle is as shown in the diagrams, in which the dashed line represents the equator.



N ← E	N ← E	W → N	W → N
sunrise, spring	sunrise, autumn	sunset, spring	sunset, autumn
$90° + \phi + \varepsilon$	$90° + \phi - \varepsilon$	$90° + \phi - \varepsilon$	$90° + \phi + \varepsilon$

At other times the equator, the ecliptic and the horizon form a spherical triangle. If Z^* is the point where the meridian through Z cuts the equator, $AZ^* = \theta$. $EZ^* = 270°$. Then $AE = 270° - \theta$ or $\theta - 270°$.

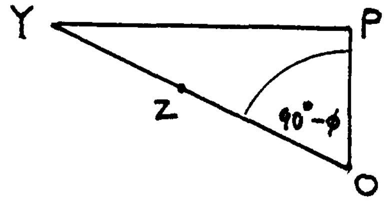


Let ζ be the angle required. Then by (III)

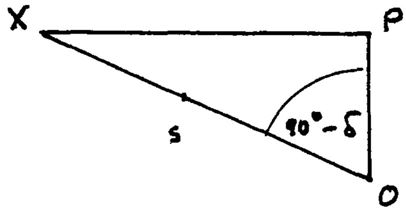
$$\cos\zeta = \cos\varepsilon.\sin\phi - \sin\varepsilon.\cos\phi.\sin\theta$$

Other possible configurations yield the same result.

To find the altitude τ of the sun at a given date and time.
Let S be the position of the sun. Let
OS and OZ cut the plane through P perpendicular to OP in
X and Y. Let OP = 1.
Let δ be the declination of the sun
at the given date. Let ψ be the rotation
of the sky since noon.



Angle YPO = 90° and YOP = ZOP = 90° - ϕ
Therefore YP = $\cot\phi$ and YO = $\operatorname{cosec}\phi$
XPO = 90° and XOP = 90° - δ
Therefore XP = $\cot\delta$ and XO = $\operatorname{cosec}\delta$.



Angle XOY = SOZ = 90° $-\tau$.
Angle YPX = ψ (because Y is south of P).
From the triangles XOY and XPY,

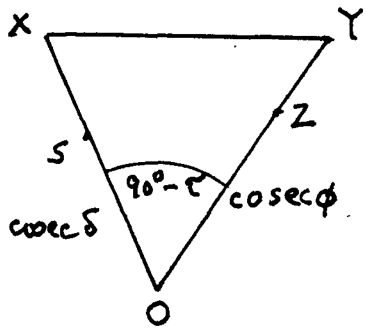
$$XY^2 = \operatorname{cosec}^2\delta + \operatorname{cosec}^2\phi - 2\operatorname{cosec}\delta.\operatorname{cosec}\phi.\sin\tau$$
$$= \cot^2\delta + \cot^2\phi - 2\cot\delta.\cot\phi.\cos\psi.$$

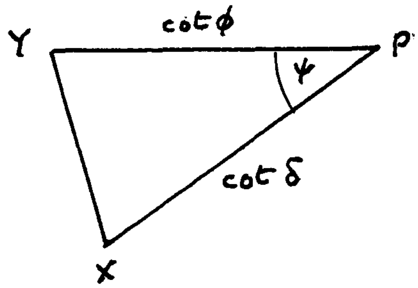
Subtract:

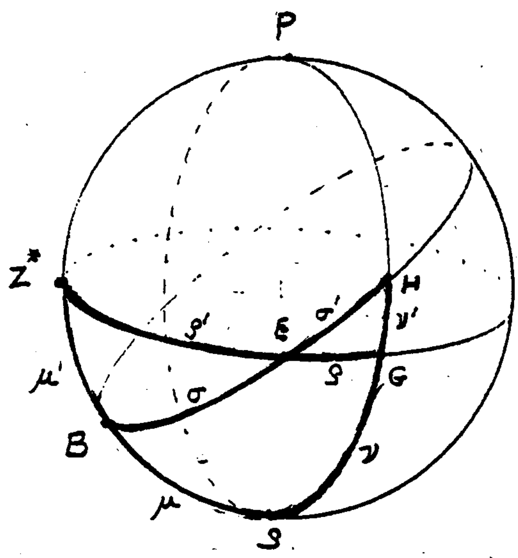
$$0 = 1 + 1 - 2\operatorname{cosec}\delta.\operatorname{cosec}\phi.\sin\tau + 2\cot\delta.\cot\phi.\cos\psi.$$

Multiply by $\frac{1}{2}\sin\phi.\sin\delta$ and rearrange:

$$\sin\tau = \sin\phi.\sin\delta + \cos\phi.\cos\delta.\cos\psi.$$







<u>The length of daylight</u>

First, suppose that the declination δ of the sun is positive. Let H be the sun rising at azimuth ζ and S the south celestial pole. Let SH cut the equator in G. The meridian through the zenith cuts the equator in Z* and the horizon in its south point B. The equator cuts the horizon in its east point E.

Let the number of degrees traversed by the sun during daylight be κ. (This is 15 times the length of daylight in hours.)

Label the segments as shown.

Then $\rho' = \sigma = \nu = 90°$, $\mu = \phi$, $\mu' = 90° - \phi$, $\nu' = \delta$, $\sigma' = 90° - \zeta$, $\rho + \rho' = \frac{1}{2}\kappa$.

Then, by (VI) inverted,

$$\tan\phi = -\cos\tfrac{1}{2}\kappa.\cot\delta$$

And (re-labelling the segments)

$$\sin\delta = \cos\zeta.\cos\phi$$

From (VII)

$$\sin\phi = -\cot\tfrac{1}{2}\kappa.\tan\zeta$$

and

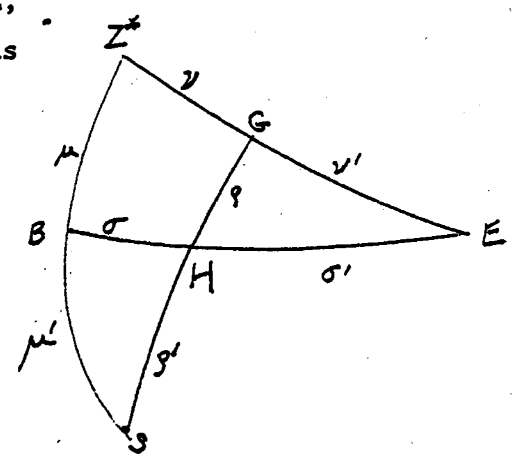
$$\cos\delta = \sin\zeta/\sin\tfrac{1}{2}\kappa$$

We now have four relations between the latitude, the length of daylight, the declination of the sun, and the azimuth at which it rises.

At the summer sostice $\delta = \varepsilon$, so the relation between the longest day and the latitude is

$$\tan\phi = -\cos\tfrac{1}{2}\kappa.\cot\varepsilon$$

Now suppose that δ is negative. B, E, etc. are as before. Label the segments as shown.



$\rho + \rho' = \sigma + \sigma' = \nu + \nu' = 90°$. $\mu' = \phi$

$\mu = 90° - \phi$. $\sigma' = \zeta - 90°$, $\sigma = 180° - \zeta$,

$\nu = \frac{1}{2}\kappa$.

From (VI)

$$\tan\phi = -\cot\delta . \cos\tfrac{1}{2}\kappa$$

and

$$\cot\tfrac{1}{2}\kappa = -\cot\zeta . \sin\phi$$

From (VII)

$$\cos\phi = -\sin\delta/(-\cos\zeta)$$

and

$$\sin\tfrac{1}{2}\kappa = \sin\zeta/\cos\delta$$

These are the same four relations as before.

At the winter solstice $\delta = -\varepsilon$, so the relation between the length of the shortest day and the latitude is

$$\tan\phi = \cos\tfrac{1}{2}\kappa . \cot\varepsilon$$

Azimuths at rising and setting

Let X be a star with declination δ rising and E the east point of the horizon. Let the meridian through X cut the equator in Q. Then in the triangle XEQ, $e = \delta$, $E = 90° - \phi$, $Q = 90°$. By the sine formula $\sin q = \sin\delta \sec\phi$. But $\zeta = 90° - q$, so

$$\cos\zeta = \sin\delta \sec\phi .$$

Let Y be the star setting and W the west point of the horizon. If the meridian through Y cuts the equator in P, $w = \delta$, $W = 90° - \phi$, $P = 90°$. Then $\sin p = \sin\delta \sec\phi$. But $\zeta = 270° + p$, so $\cos\zeta = \sin p$, and again

$$\cos\zeta = \sin\delta \sec\phi.$$

Of the two angles whose cosine is $\sin\delta \sec\phi$, one gives the azimuth at rising, the other at setting.

The effect of precession on declination and right ascension

A dot denotes rate of change.

$\sin\delta = \sin\beta\cos\varepsilon + \cos\beta\sin\lambda\sin\varepsilon$, so $\dot{\delta}\cos\delta = \dot{\lambda}\cos\beta\cos\lambda\sin\varepsilon = \dot{\lambda}\cos\alpha\cos\delta\sin\varepsilon$

Then $\dot{\delta} = \dot{\lambda}\cos\alpha\sin\varepsilon$.

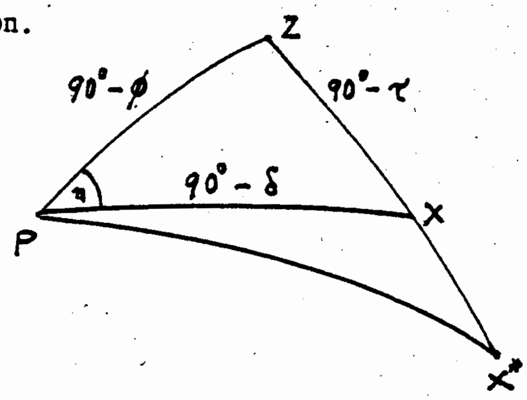
$\cos\delta\cos\alpha = \cos\beta\cos\lambda$, so $\dot{\delta}\sin\delta\cos\alpha + \dot{\alpha}\cos\delta\sin\alpha = \dot{\lambda}\cos\beta\sin\lambda$.

$$\begin{aligned}\text{Then } \dot{\alpha}\cos\delta\sin\alpha &= \dot{\lambda}(\cos\beta\sin\lambda - \sin\delta\cos^2\alpha\sin\varepsilon).\\ &= \dot{\lambda}(\cos\delta\sin\alpha\cos\varepsilon + \sin\delta\sin\varepsilon - \sin\delta\cos^2\alpha\sin\varepsilon).\\ &= \dot{\lambda}(\cos\delta\sin\alpha\cos\varepsilon + \sin\delta\sin^2\alpha\sin\varepsilon), \text{ so}\end{aligned}$$

$$\dot{\alpha} = \dot{\lambda}(\cos\varepsilon + \tan\delta\sin\alpha\sin\varepsilon).$$

To find the effect on equatorial coordinates of altering the altitude of a body without altering its azimuth.

This is useful because terrestrial parallax alters the observed altitude of a body without altering its azimuth. So does refraction.



Let X have coordinates η, δ and τ.
Let X* have altitude τ^* and the same azimuth as X.
Denote its equatorial coordinates by η^* and δ^*.
$PZ = 90° - \phi$, $ZX = 90° - \tau$, $ZPX = \eta$, $PX = 90° - \delta$.
By (IV)

$$\tan\eta = \frac{\sin Z}{\cos\phi.\tan\tau - \sin\phi.\cos Z}$$

Therefore

$$\sin\phi.\cos Z = \cos\phi.\tan\tau - \sin Z.\cot\eta.$$

Similarly

$$\sin\phi.\cos Z = \cos\phi.\tan\tau^* - \sin Z.\cot\eta^*.$$

Therefore

$$\cos\phi(\tan\tau - \tan\tau^*) = \sin Z(\cot\eta - \cot\eta^*)$$

$$= \frac{\sin\eta.\cos\delta}{\cos\tau}(\cot\eta - \cot\eta^*) \qquad \text{by (I)}$$

This gives η^*.

By (II) $\sin\delta = \sin\phi.\sin\tau + \cos\phi.\cos\tau,\cos Z$

$$\sin\delta^* = \sin\phi.\sin\tau^* + \cos\phi.\cos\tau^*.\cos Z$$

$$= \sin\phi.\sin\tau^* + \cos\tau^*(\sin\delta - \sin\phi.\sin\tau)/\cos\tau.$$

This gives δ^*.
The change in α is the negative of the change in η.

Location of a point with respect to a circle

If we know the angles subtended by points P, Q, and R on a circle at the centre and at a point E we can locate E.

Let the radius of the circle be r and its centre O. Let the angles subtended at E by PQ, QR, and RP be α, β and γ, all measured in the same direction, so that $\alpha + \beta + \gamma = 360°$. Let $\lambda = POQ - \alpha$, $\mu = QOR - \beta$, and $\nu = ROP - \gamma$. These angles are known. Let angles OEQ = θ and EQO = ϕ. Let OE cut the circle in A and B. Drop a perpendicular OX from O to EQ.

$QOA = OEQ + EQO = \theta + \phi$

$POA = QOA - POQ = (\theta + \phi) - (\alpha + \lambda)$

$PEA = QEA - QEP = \theta - \alpha$

$EPO = POA - PEA = \phi - \lambda.$

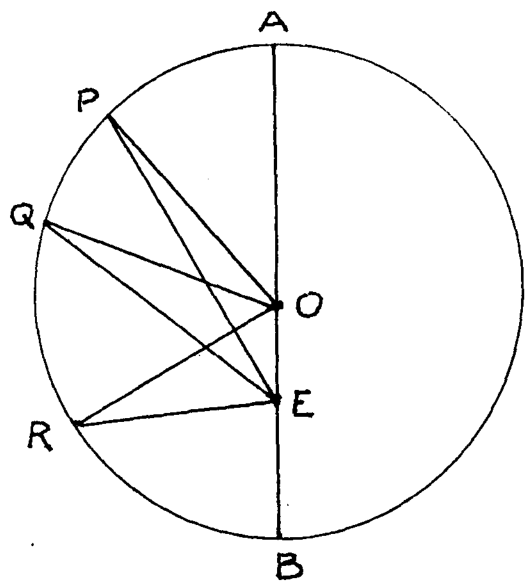
From the triangle OPE.

$OE\sin(\theta - \alpha) = OP\sin(\phi - \lambda)$

$OE(\sin\theta\cos\alpha - \cos\theta\sin\alpha) = r(\sin\phi\cos\lambda - \cos\phi\sin\lambda)$

$OX\cos\alpha - XE\sin\alpha = OX\cos\lambda - QX\sin\lambda$

$$OX(\cos\alpha - \cos\lambda) = XE\sin\alpha - QX\sin\lambda \qquad (1)$$



$ROA = QOA + ROQ = \theta + \phi + \beta + \mu$

$REA = REQ + QEO = \theta + \beta.$

$ERO = ROA - REA = \phi + \mu.$

From the triangle REO,

$OE\sin(\theta + \beta) = r\sin(\phi + \mu)$

$OE(\sin\theta\cos\beta + \cos\theta\sin\beta) = r(\sin\phi\cos\mu + \cos\phi\sin\mu)$

$OX\cos\beta + XE\sin\beta = OX\cos\mu + QX\sin\mu$

$$OX(\cos\beta - \cos\mu) = -XE\sin\beta + QX\sin\mu. \qquad (2)$$

Multiply (1) by $\sin\mu$ and (2) by $\sin\lambda$ and add. Then

$$aOX = bXE$$

where $\quad a = \cos\alpha\sin\mu + \cos\beta\sin\lambda + \sin\nu$

(because $-\sin\mu\cos\lambda - \sin\lambda\cos\mu = -\sin(\lambda + \mu) = \sin y$.)

$$b = \sin\alpha\sin\mu - \sin\beta\sin\lambda.$$

Multiply (1) by $\sin\beta$ and (2) by $\sin\alpha$ and add. Then

$$cOX = bQX$$

where $\quad c = -\sin\gamma - \sin\alpha\cos\mu - \sin\beta\cos\lambda$

(because $\sin\beta\cos\alpha + \sin\alpha\cos\beta = \sin(\alpha + \beta) = -\sin\gamma$.)

Set $p = a^2 + b^2 = \cos^2\alpha\sin^2\mu + \cos^2\beta\sin^2\lambda + \sin^2\nu$

$\quad + 2\cos\alpha\cos\beta\sin\lambda\sin\mu + 2\cos\alpha\sin\mu\sin\nu + 2\cos\beta\sin\nu\sin\lambda$

$\quad + \sin^2\alpha\sin^2\mu + \sin^2\beta\sin^2\lambda - 2\sin\alpha\sin\beta\sin\lambda\sin\mu$

$= \sin^2\lambda + \sin^2\mu + \sin^2\nu + 2\cos\alpha\sin\mu\sin\nu + 2\cos\beta\sin\nu\sin\lambda + 2\cos\gamma\sin\lambda\sin\mu$

(because $\cos\alpha\cos\beta - \sin\alpha\sin\beta = \cos(\alpha + \beta) = \cos\gamma$)

Set $q = b^2 + c^2 = \sin^2\alpha\sin^2\mu + \sin^2\beta\sin^2\lambda - 2\sin\alpha\sin\beta\sin\lambda\sin\mu$

$\quad + \sin^2\gamma + \sin^2\alpha\cos^2\mu + \sin^2\beta\cos^2\lambda$

$\quad + 2\sin\gamma\sin\alpha\cos\mu + 2\sin\beta\sin\gamma\cos\lambda + 2\sin\alpha\sin\beta\cos\lambda\cos\mu$

$= \sin^2\alpha + \sin^2\beta + \sin^2\gamma + 2\cos\lambda\sin\beta\sin\gamma + 2\cos\mu\sin\gamma\sin\alpha + 2\cos\nu\sin\alpha\sin\beta$

(because $\cos\lambda\cos\mu - \sin\lambda\sin\mu = \cos(\lambda + \mu) = \cos\nu$).

$OE^2 = OX^2 + XE^2 = OX^2(1 + a^2/b^2) = pOX^2/b^2$

$r^2 = OX^2 + QX^2 = OX^2(1 + c^2/b^2) + qOX^2/b^2$.

Therefore $OE^2/r^2 = p/q$. This determines the distance of E from O: E lies on a circle of this radius with centre O.

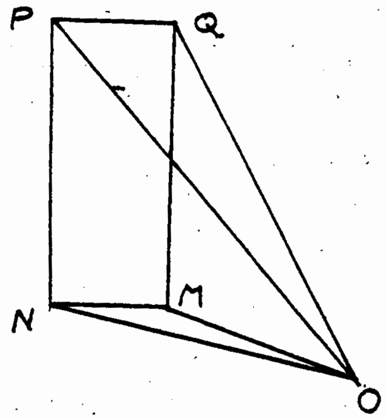
$\tan QEO = OX/OE = b/a$. This fixes the direction of the line AB. E must be on of the two points where this line cuts the circle just mentioned.

In any particular case we can calculate the angle subtended by PQ at each of these points to decide which is the one wanted.

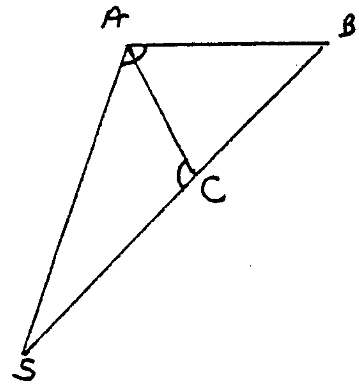
Orientation

O is the observer. OMQ is a vertical plane with azimuth ζ. P is the north celestial pole or any point on the line from O to the pole. PQ is the perpendicular from P to the vertical plane. PQMN is a rectangle. Then angle MON $= \zeta$. Let angle POQ (the angle between OP and the vertical plane) be ξ. Angle PON $= \phi$.

$$\sin\zeta = MN/ON = PQ/ON = (PQ/PO).(PO/ON) = \sin\xi \sec\phi.$$



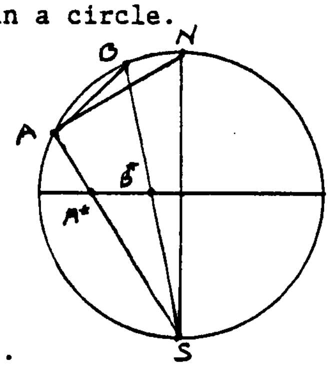
The astrolabe



The commonest form of astrolabe projects the celestial sphere onto its equatorial plane from the south pole. Every circle on the sphere projects into a circle. We show this as follows.

The lines from a point S to a circle form a cone. Let AB be the diameter of the circle through the foot of the perpendicular from S to the plane of the circle.

Imagine the plane of the circle to rotate about a line through A perpendicular to the plane SAB. It will cut the cone in an ellipse whose eccentricity at first increases as the plane rotates and then decreases, becoming zero again (by symmetry) when, if it cuts SB in C, the angle ACS = SAB. Every plane parallel to this will also cut the cone in a circle.



Now let S be the south pole (and N the north pole) of the celestial sphere.

Given any circle on the sphere, let the plane through N, S and the centre of the circle cut the circle in A and B. Let A* and B* be the projections of A and B.

B*A*S = 90° − ASN = ANS = ABS (angles in the same segment).

Therefore the cone formed by S and the given circle is cut by the plane through A* and B* perpendicular to the plane SAB in a circle. This is the projection made by the astrolabe.