An Interesting Property of the Equant

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Early astronomers were well aware of the fact that the speed of the planets varies as the planets move around the zodiac. This zodiacal anomaly, a departure from the perfect uniform motion expected of a celestial body according to many philosophers, was modeled by the early astronomers in several ways. For the Sun and the Moon (neglecting for now the second lunar anomaly), the simplest models used either an eccentric deferent with eccentricity e or an epicycle of radius r.

In the eccentric deferent model, the planet rotates around a deferent circle of radius R at a uniform speed as seen from the center of the circle. The Earth, which is of course taken as the center of the universe, is displaced a distance e from the center of the deferent. The effect of this shift is that as seen from the Earth the speed of the planet is slowest when the planet is on the extension of the line from Earth to the center of the deferent (apogee), and fastest in the opposite direction, when the planet is on the extension of the line from the center of the deferent to the Earth (perigee). The planet moves at its mean speed when it is normal to the apsidal line as seen from the Earth, which means that it is more than 90° from apogee as seen from the center of the deferent.

In the epicycle model, the planet rotates clockwise[1] around a (smaller) epicycle circle of radius r at a uniform speed as seen from the center of the epicycle, and the center of the epicycle moves at uniform speed around a deferent circle of radius R, which is centered on the Earth. In this case, the minimum speed of the planet occurs when the planet is at the apogee of the epicycle, i.e. when it is as far as possible from the Earth, and its maximum speed occurs when the planet is at the perigee of the epicycle, i.e. when it is as near as possible to the Earth. The planet once again moves at its mean speed when it is normal to the apsidal line as seen from the Earth, which of course means that the epicycle center is more than 90° from the direction of apogee as seen from Earth.

In *Almagest* 3.3, regarding the Sun, and again in *Almagest* 4.5 for the Moon, Ptolemy (*ca.* A.D. 150) explains in great detail the geometrical equivalence of these models.[2] This is seen most easily in Figure 1, which shows the two models superimposed, and the equivalence follows immediately from the elementary geometrical properties of the parallelogram. The equivalence of the models is further demonstrated in *Almagest* 3.5 by numerical examples, where Ptolemy essentially shows that the equation of center for the eccentric model is

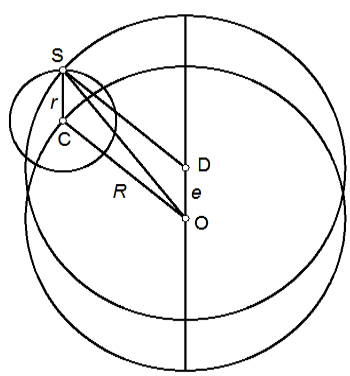

Figure 1. In the epicycle mode the Earth is at the center O of a deferent circle of radius R, an epicycle of radius r is centered at C, and the planet is at S on the epicycle. In the eccentric model the Earth is displaced a distance e from the center D of the eccentric deferent of radius R, and the planet is at S on the deferent. If $r = e$, then $ODSC$ is a parallelogram and the position of S as seen from O is the same in both models.

[1] This clockwise motion on the epicycle when modeling the zodiacal anomaly should not, of course, be confused with the counterclockwise motion of the planets on their epicycles when modeling the solar anomaly, which is responsible for retrograde motion.
[2] G. J. Toomer, *Ptolemy's Almagest* (1984).

$$\tan q = \frac{-e\sin\alpha}{R + e\cos\alpha}$$

and for the epicycle model it is

$$\tan q = \frac{r\sin\gamma}{R + r\cos\gamma},$$

so the equations of center are the same when $e = r$ and $\alpha = -\gamma$ (the sign arising from the opposite direction of increase for the angles). We know from Theon of Smyrna,[3] writing perhaps a generation earlier than Ptolemy, that the equivalence of eccentric and epicycle models was broadly understood among early astronomers, at least as far back as Hipparchus (*ca.* 130 B.C.), and we furthermore know from remarks by Ptolemy in *Almagest* 12.1 that the equivalence was very likely understood as far back as the time of Apollonius of Perge (*ca.* 200 B.C.).

Beyond these models, however, there is very good reason to believe that an additional model was used for the Sun and the Moon by Greco-Roman astronomers, most likely between the time of Hipparchus and Ptolemy.[4] The modern name for that model is the concentric equant, and it was used exclusively for the Sun and Moon in texts from the fifth through seventh centuries A.D. from ancient India.[5] For the concentric equant, the Earth is at the center of a deferent circle which is the orbit of the Sun or Moon, but the motion of the luminary on the deferent is seen as uniform not from the Earth, but from a point, the equant, displaced a distance *e* from the center of the deferent (see Figure 2). The speed of the planet as seen from the earth is slowest in the direction of the equant, fastest in the direction opposite the equant, and it has its mean value when the planet is 90° from the direction of apogee as seen from Earth.

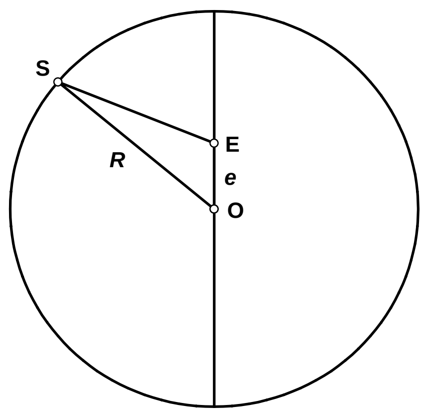

Figure 2. In the concentric equant model the Earth is at the center *O* of a deferent of radius *R* and the planet is at *S* on the deferent. The motion of *S* is uniform as seem from the equant point *E*, which is a distance *e* from *O*.

The idea that these ancient Indian texts are ultimately of Greco-Roman origin, and from the time between Hipparchus and Ptolemy, dates from the very first investigations by Western scholars in the 1800's.[6] In the second half of the 20th century the most prominent champions of the idea, and the scholars who did the most to document and elaborate it, were Pingree[7] and van der Waerden,[8] and so it seems appropriate to refer to the idea as the Pingree – van der Waerden (PvdW) hypothesis. The principal basis of the argument is that almost all of the astronomical features in the early texts are significantly less developed than those we find in the *Almagest*.

[3] R. & D. Lawlor, *Theon of Smyrna: Mathematics Useful for Understanding Plato Or, Pythagorean Arithmetic, Music, Astronomy, Spiritual Disciplines* (1978).
[4] David Pingree, "History of Mathematical astronomy in India", *Dictionary of Scientific Biography*, **15** (1978), 533-633.
[5] K. S. Shukla, "Use of Hypotenuse in the Computation of the Equation of Center under the Epicyclic Theory in the School of Aryabhata I ???", *Indian Journal of History of Science,* **8** (1973) 43-57.
[6] E. Burgess and W. D. Whitney, "Translation of the Surya Siddhanta", *Journal of the American Oriental Society*, (1858) 141-498, and references therein.
[7] David Pingree, "On the Greek Origin of the Indian Planetary Model Employing a Double Epicycle", *Journal for the history of astronomy*, ii (1971), 80-85; D. Pingree, "The Recovery of Early Greek astronomy from India", *Journal for the history of astronomy*, **7** (1976), 109-123; D. Pingree, *ibid.* (ref. 4), and references therein.
[8] B. L. van der Waerden, "The heliocentric system in greek, persian, and indian astronomy", in *From deferent to equant: a volume of studies in the history of science in the ancient and medieval near east in honor of E. S. Kennedy*, Annals of the new york academy of sciences, **500** (1987), 525-546, and references therein.

The temporal coexistence of the concentric equant and the eccentric/epicycle models and the extensive surviving discussion of the equivalence of those models immediately begs the questions (a) can we extend the equivalence theorem to include all three models, and (b) is there any evidence that the ancient astronomers were aware of such an equivalence? The answer to (a) is a definite yes, but with an interesting twist, and the answer to (b) is also yes, at least in the framework of the PvdW hypothesis.

The geometrical equivalence of the concentric equant and eccentric models is illustrated in Figure 3.

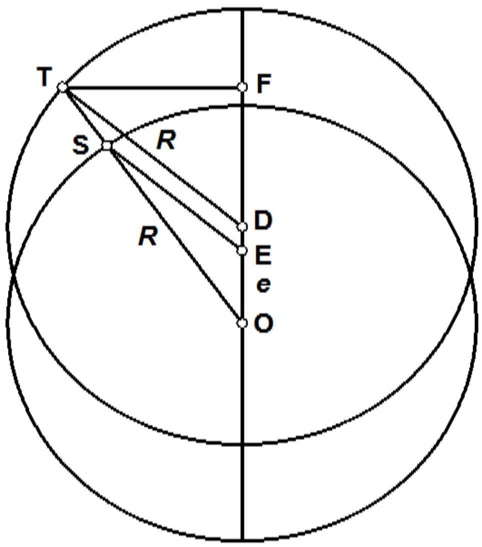

Figure 3. The concentric equant has the Earth at *O*, the planet at *S* on a deferent of radius *R*, and the equant at *E*. The equivalent simple eccentric has an eccentric deferent also of radius *R* but centered at *D*, and the planet is now at *T*. Since *ES* and *DT* are parallel both models have the same mean centrum (angles *FDT* and *DES*), and since *OST* is a straight line, the *S* and *T* have the same position as seen from the Earth and so both models have the same true centrum (angles *EOS* and *DOT*).

The models will be equivalent if and only if they predict the same value of the true centrum given the same value for the mean centrum. It is clear from the figure that this will in general be the case if and only if the eccentricity in the eccentric model is not constant but instead oscillates in a well-defined way about the value of the eccentricity in the equant model. This can be proved analytically by considering two cases:

(1) an eccentric model with eccentricity e', and
(2) a concentric equant model with eccentricity *e*.

For the eccentric the equation of center is given by

$$\tan q_1(e',\alpha) = \frac{-e'\sin\alpha}{R+e'\cos\alpha},$$

while for the concentric equant we have

$$\tan q_2(e,\alpha) = \frac{-e\sin\alpha}{\sqrt{R^2-e^2\sin^2\alpha}-e\cos\alpha},$$

where α is the mean centrum. These models are equivalent when

$$q_1(e'(\alpha),\alpha) = q_2(e,\alpha)\,,$$

so the required function $e'(\alpha)$ is given by

$$\begin{aligned} e'(\alpha) &= \frac{-R\tan q_2(e,\alpha)}{\sin\alpha + \cos\alpha\tan q_2(e,\alpha)} \\ &= \frac{eR}{\sqrt{R - e^2\sin^2\alpha} - e\cos\alpha} \end{aligned}$$

It is clear that

$$\frac{1}{1+\dfrac{e}{R}} \le \frac{e'}{e} \le \frac{1}{1-\dfrac{e}{R}}$$

and that e'/e varies smoothly from its maximum at apogee to its minimum at perigee (see Figure 4). This then answers question (a) in the affirmative, and defines how the eccentricity of a simple eccentric model must vary in order to give the same equation of center as the concentric equant for a given mean centrum. Since the simple eccentric model is equivalent to a simple epicycle model, it immediately follows that the concentric equant is also equivalent to an epicycle model with a radius r' varying in the same way.

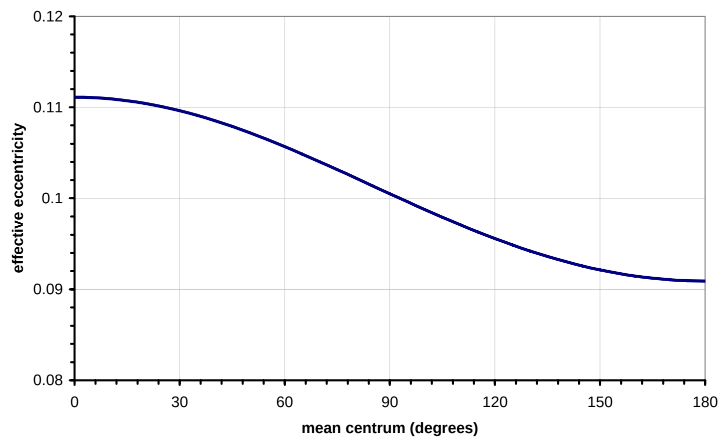

Figure 4. The effective eccentricity e'/R of the simple eccentric model that is equivalent to a concentric equant with $e/R = 0.1$. Clearly e' is maximum when the mean centrum α is zero, or at apogee, and e' is minimum at perigee.

Of course question (a) would be of only academic interest unless we can also answer affirmatively question (b), and clearly establish an historical background for believing that the extension of the equivalence to the concentric equant was known to ancient astronomers. In fact, the point is demonstrated explicitly in Bhaskara's *Mahabhaskariya*[9] (A.D. 629) which is a commentary on the *Aryabhatiya*[10] (A.D. 499), the primary text of Aryabhata. Bhaskara explains the equivalence of the concentric equant and an oscillating eccentric model by directly computing one from the other, as follows:[11]

[9] K. S. Shukla, *Mahabhaskariya of Bhaskara I* (1960).
[10] K. S. Shukla, *Aryabhatiya of Aryabhata* (1976).
[11] David Pingree, "Concentric with Equant", *Archives Internationales d'Histoire des Sciences,* **24** (1974) 26-28.

IV 19. Subtract (the Sine of) the final equation from (the Sine of) the koti or again add it, depending on the quadrant; the square-root of the sum of the square of that and the square of the bahu is the hypotenuse.
IV 20. Multiply (the Sine of) the final equation by the hypotenuse (and) divide (the product) by the Radius; add (the quotient) to or subtract it from the previous koti (repeatedly) until the hypotenuse is equal (to the hypotenuse obtained in the immediately preceding computation).
IV 21. Multiply the Radius by the Sine of the bahu (and) divide (the product) by the (final) hypotenuse. Add the arc (corresponding to that quotient) to (the longitude of) the apogee according to the quadrant of the argument.

However terse this may appear to us, it is actually rather verbose in comparison to many early Indian astronomical texts. In any event, the execution of the algorithm is as follows (see again Figure 3):

Let $OS = DT = R$ and adjust point D so that DT is parallel to OS and triangles OES and ODT are similar. The mean centrum α (angles DES and FDT) and the eccentricity $EO = e$ of the concentric equant are given. The algorithm finds OD and $OT = h$ = 'the hypotenuse', and uses those to compute the true centrum (angle EOS). First, drop a perpendicular line from T to a point F on the apsidal line.

Step 1. (verse 19) assume $OD = e$

Step 2. then $OF = OD + R\cos(\alpha)$ and $TF = R\sin(\alpha)$. Here $R\cos(\alpha)$ and $R\sin(\alpha)$ are the Sine of the koti and the Sine of the bahu, respectively.

Step 3. then $OT = \sqrt{OF^2 + TF^2} = h$

Step 4. (verse 20) by similar triangles $\dfrac{OT}{OD} = \dfrac{R}{e}$, so we have a new estimate for $OD = eh / R$.

Step 5. go to step 2 with the new estimate of OD and recompute OF, TF, and h. When h stops changing, go to step 6.

Step 6. (verse 21) compute angle $DOT = c = \arcsin(R\sin\alpha / h)$ which, added to the longitude of apogee, gives the longitude of the planet.

The algorithm solves $h = \sqrt{(eh + R\cos\alpha)^2 + R^2\sin^2\alpha}$ by iteration beginning with $h = R$ as the first trial value. The angle computed in Step 6 will, of course, be the same value you could have gotten directly from $c = \alpha + q = \alpha + \arcsin(-e / R\sin\alpha)$ with a much simpler calculation (dropping a perpendicular from O to the extension of SE), so it is not clear why the iterative solution was used.

Verses IV 9-12 in Bhaskara's text give a similar solution employing an epicycle of varying radius. Invoking the PvdW hypothesis, we conclude that some Greco-Roman astronomers not only knew and used the concentric equant for the Sun and Moon, but they also understood that it was equivalent to an eccentric model with oscillating eccentricity and an epicycle model with oscillating radius.

The modern history of the concentric equant is interesting in itself. In 1952 van der Waerden noticed that the Tamil rules (*ca.* A.D. 1300) for computing solar longitude, based on the 248 day anomalistic cycle, were explained better by a concentric equant than by an Hipparchan eccentric model.[12] In 1956, van der Waerden's student Krishna Rav showed that the Tamil lunar longitudes computed using the same 248 day cycle were also explained better by a concentric equant model than by either an Hipparchan model or Babylonian System A or System B schemes for the lunar motion.[13]

[12] B. L. van der Waerden, "Die Bewegung der Sonne nach Griechischen und Indischen Tafeln", *Bayer. Akad. Wiss. Munchen* 1952, math.-nat. K1., 219.
[13] I. V. M. Krishna Rav, "The Motion of the Moon in Tamil Astronomy", *Centaurus*, **4** (1956) 198-220.

However, in 1956 in the paper in *Centaurus* immediately following Krishna Rav's, van der Waerden changed his mind.[14] He claimed that since there was no known tradition of a concentric equant model in either Greek or Indian astronomy, it would be better to assume that the Indians were in fact using a conventional Hipparchan eccentric model, but were computing the equation of center by approximation. Thus, while $\sin q = -e/R \sin\alpha$ is exact for the concentric equant and is indeed used exclusively in Indian astronomy to compute the equation of center for the Sun and Moon, it is also a good approximation to the Hipparchan equation of center for small *e*, since

$$\sin q = -e/R\sin(\alpha + q) \simeq -e/R\sin\alpha + O(e^2/R^2)\ .$$

Thus van der Waerden concluded that the agreement of the approximation with the concentric equant was accidental.

However, in 1974 Pingree pointed out that there is indeed an explicit discussion of the concentric equant in Indian astronomy, the very commentary on Aryabhata by Bhaskara discussed above, thus contradicting the premise of van der Waerden's doubt.[15] Since Pingree, van der Waerden, and virtually all other western scholars agree that these Indian texts represent a tradition derived from much older Greco-Roman sources, it appears that van der Waerden's and Krishna Rav's original conclusions are indeed correct.

There is one other possible reflection of the concentric equant in ancient Greek astronomy. In Book 5 of the *Almagest* Ptolemy resolves the discrepancies between the simple Hipparchan lunar model and lunar elongations at quadrature by adding a complication to the model that bears a striking resemblance to the concentric equant. Indeed, Ptolemy's lunar model is a concentric equant as discussed above, with the modification that the Earth is positioned not at the center of the deferent but at the equant point. This is the earliest point in the *Almagest* that Ptolemy employs a deviation from uniform circular motion, and he does so here silently, so we have no information from him on the origin of the model. Ptolemy goes further by adjusting the position of the apsidal line of the lunar epicycle, but that has no bearing on the modified concentric equant construction he uses. It might, however, be worth noting that according to one plausible reading of Ptolemy's earliest commentator, the shift of the epicycle's apogee is in fact the *only* contribution by Ptolemy to the full lunar model.[16] Such an interpretation is, of course, not inconsistent with the present discussion.

It is also at least possible that the concentric equant was at some point used for not only the Sun and Moon, but that with an added epicycle it was also used for planetary motion. We have no textual evidence for this, but the textual evidence that the simple eccentric plus epicycle model was ever used for planetary motion is also very sparse, so both cases are essentially on the same footing. Such joint models exhibit both zodiacal and solar anomalies, the latter being responsible for retrograde motion. In addition, a concentric equant plus epicycle model has a great practical advantage over the eccentric plus epicycle model in that the two anomalies are not coupled, so computation is as easy as a couple of table look-ups, and no complicated decoupling interpolation, such as the scheme Ptolemy provides in the *Almagest,* is required for computation using tables. It turns out, however, that such a model is no better or worse for explaining the observed phenomena than an eccentric plus epicycle model, although the models fail in virtually opposite ways, so for example when one model produces retrograde arcs that are too small, the other produces arcs too large, and *vice versa* (see Figure 5).

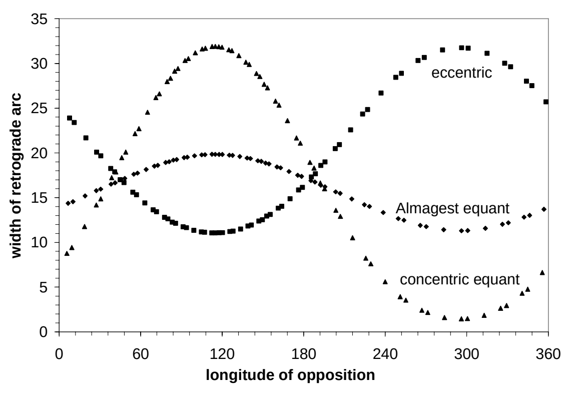

Figure 5. The width of retrograde arcs for Mars is shown as a function of zodiacal longitude. The apogee of Mars is around 115°, so the simple eccentric model gives arcs of maximum width at apogee and minimum width at perigee, while the concentric equant gives just the opposite pattern. The much weaker variation predicted by the *Almagest* equant is close to actual observation.

[14] B. L. van der Waerden, "Tamil Astronomy", *Centaurus*, **4** (1956) 221-234.
[15] David Pingree, *ibid*. (ref. 11).
[16] Alexander Jones, *Ptolemy's first commentator.* Philadelphia, 1990. Transactions of the American Philosophical Society, 80.7. 62 pp.

At some point between the times of Hipparchus and Ptolemy astronomers realized that a more sophisticated model, the equant, was required to account for the observed phenomena. One way to think of the equant is to start from an eccentric deferent model, and to displace the center of uniform motion toward the apogee, and hence away from the Earth, by an amount equal to the distance *e* between Earth and the center of the deferent. But an equivalent view is to start with a concentric equant and displace the Earth away from the equant. Either way, if any ancient analyst had noticed the pattern of mirror image failing of the simplest models, then it is tempting to muse that at least one factor leading to the *Almagest* equant was a consideration of the sort shown schematically in Figure 6, where the full equant is seen as a sort of merging of an eccentric deferent and a concentric equant.

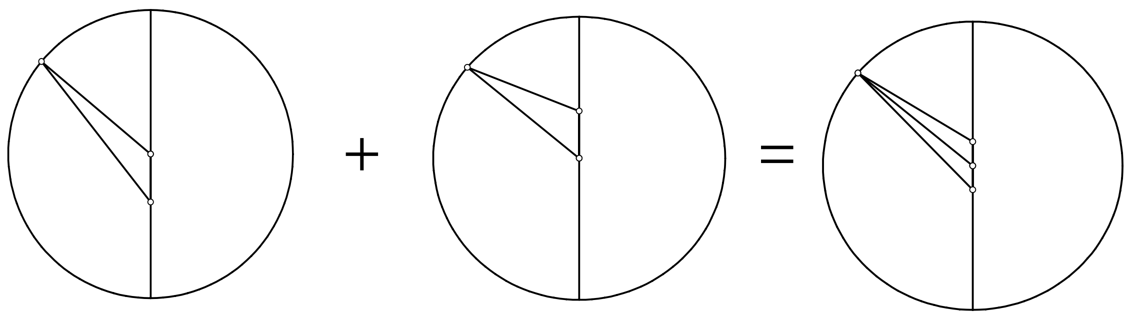

Figure 6. A schematic representation of how someone might have thought to combine the simple eccentric and concentric equant models into the *Almagest* equant.

Ptolemy, of course, uses the equant extensively in the *Almagest* to account for the zodiacal motion of Venus, Mars, Jupiter, and Saturn, and he gives detailed derivations of the equant model parameters based on empirical data for each of those planets, but he gives virtually no historical information on the development of the equant, and no other author even mentions its existence until Islamic astronomers began criticizing its philosophical shortcomings (nonuniform motion as seen from the center of the deferent and the impossibility of implementing such motion with rigid rotating celestial spheres). Several paths to the equant have been proposed, none of them mutually exclusive of the others, but the simple fact is that we cannot be sure when the equant was invented or how long it took, who invented it, or even what empirical data triggered the invention.[17] So whether or not anyone ever thought about planetary models along this line is of course a matter of speculation, but hardly more so than all other proposed histories of the equant.

In this context it is interesting to note that while all early Indian texts use the concentric equant to account for the zodiacal anomaly in both solar and lunar models, they use more complicated constructs to account for the planets, which exhibit both zodiacal and solar anomalies. Despite being effectively disguised by a remarkable series of approximations, the underlying mathematical basis of these Indian models is precisely the *Almagest* equant.[18] In particular, the longitude in the Indian models is computed by means of a sequence of steps involving the concentric equant for the zodiacal anomaly, using $\sin q = -e / R \sin \alpha$, and the usual epicycle equation

$$\tan p = \frac{r \sin \gamma}{R + r \cos \gamma}$$

[17] James Evans, "On the function and probable origin of Ptolemy's equant", *American journal of physics,* **52** (1984), 1080-9; Noel Swerdlow, "The empirical foundations of Ptolemy's planetary theory", *Journal for the history of astronomy*, **35** (2004), 249-71; Alexander Jones, "A Route to the ancient discovery of non-uniform planetary motion", *Journal for the history of astronomy*, **35** (2004); Dennis W. Duke, "Comment on the Origin of the Equant papers by Evans, Swerdlow, and Jones", *Journal for the History of Astronomy* **36** (2005) 1-6.

[18] B. L. van der Waerden, "Ausgleichspunkt, 'methode der perser', und indische planetenrechnung", *Archive for history of exact sciences*, 1 (1961), 107-121; Dennis W. Duke, "The equant in India: the mathematical basis of Indian planetary models", *Archive for History of Exact Sciences*, **59** (2005) 563-576.

for the solar anomaly. The development of this approximation scheme, itself an especially brilliant achievement in applied mathematics, might well be expected from someone familiar with the concentric equant, but of course by no means proves that case.

The equivalence of the concentric equant to an eccentric with oscillating eccentricity and an epicycle with oscillating radius can be easily extended to the *Almagest* equant. The discussion parallels that given above for the concentric equant, except for the *Almagest* equant one has

$$\tan q_2(e,\alpha) = \frac{-2e\sin\alpha}{\sqrt{R^2 - e^2\sin^2\alpha} + e\cos\alpha}$$

and so

$$e'(\alpha) = \frac{2eR}{\sqrt{R^2 - e^2\sin^2\alpha} - e\cos\alpha}\ .$$

In this case e'/e is bounded by

$$\frac{1}{1+\frac{e}{R}} \le \frac{e'}{2e} \le \frac{1}{1-\frac{e}{R}}$$

and so the variation is smaller than in the case of the concentric equant.

The fact that such an equivalence scheme for the equant is never mentioned in any Greco-Roman text is hardly surprising. Since only Ptolemy mentions the equant, and since we know that Ptolemy, and according to Theon of Smyrna, earlier astronomers, were developing a cosmology of the heavens in terms of physical spheres, it is certainly not unreasonable to suppose that even if he knew about this oscillating radius/eccentricity view of the equant, Ptolemy would see no reason to mention it since it conflicts so starkly with his view of cosmological reality.

However, one very curious feature of most of the Indian schemes is the use of pulsating values for *e* and *r*. One might suppose that these pulsating values somehow reflect an earlier knowledge of the relationship of the equant to oscillating *e* and *r* values, except for the facts that (a) the Indian schemes have *two* maxima and minima per rotation instead of the single maximum and minimum that we find in the equivalence with the equant, and (b) the Indian schemes already incorporate the equant, and the oscillating eccentricity arises from the equivalent simple eccentric model. Altogether this suggests that someone was tinkering, perhaps in order to improve some perceived fault in the model. Whether this tinkering was done originally in Greco-Roman times or later in India seems impossible to say in the absence of any further evidence.