MATH 728D: Machine Learning Homework #1: Linear Algebra

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This homework is intended as exercises for you to familiarize yourself with the course material. It will not be collected or graded. If you have questions about the exercises, these can be answered through email or at office hours

1 Householder Transformations

A Householder transformation, to be applied to column j of a matrix A, is constructed follows:

1. Set v to column j of A;
2. Zero out entries 1 through $j-1$ of v;
3. Add to v_j the value $\text{sign}(v_j) * ||v||$;
4. Define matrix $Q_j = I - \frac{2}{v'v} vv'$;
5. Multiplying $Q_j * A$ will set column j to upper triangular form;

The QR factorization can be computed by successively multiplying matrix A by Q_1, Q_2, ..., Q_k.

In the following, assume that v is any (nonzero) vector, and that Q is the corresponding Householder transformation matrix $Q = I - \frac{2}{v'v} vv'$:

1. Show that Q is symmetric;
2. Show that Q is orthogonal;
3. Show that $Q\,v = -v$;

2 The QR Factorization

The QR factorization of a matrix A can be carried out by applying a sequence Q_1, Q_2, ..., Q_k of Householder transformations, so that $Q_1 * Q_2 * ... * Q_k * A = R$ where R is an upper triangular matrix.

1. Based on this fact, what is the Q factor for A?
2. How do you know that Q is an orthogonal matrix?
3. The value k counts the number of Householder transformations we must apply. The j-th transformation reduces column j to upper triangular form. What is the typical value of k when A is an $n \times n$ matrix? What happens if A is an $m \times n$ matrix?

3 The LR Factorization

The process of Gauss elimination is. equivalent to computing the following factorization of the matrix A:

$$A = P' * L * R$$

where P is a permutation matrix, L is unit lower triangular, and R is upper triangular.

1. Describe how this factorization can be used to solve a linear system $A * x = b$;
2. The inverse of a triangular matrix is easy to compute. Assuming this is so, what is a formula for the inverse of a matrix A for which we have computed the LR factorization?

4 The SVD Factorization

The singular value decomposition factors a matrix as follows:

$$A = U * S * V'$$

where U and V are orthogonal matrices, and S is a diagonal matrix. The diagonal entries of D are called *singular values*; a typical entry is symbolized by σ_i.

1. Show that the columns of U are eigenvectors of AA';
2. Show that the values σ_i^2 are (some of the) eigenvalues of AA' and $A'A$;
3. Suppose that A is symmetric and positive definite, so that it has a full set of eigenvectors X and nonnegative eigenvalues Λ. Write a formula for the singular value decomposition of A;
4. Suppose that A is a square, invertible matrix. Write a formula for the inverse of A in terms of the singular value decomposition;