The Death Map

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https://people.sc.fsu.edu/∼jburkardt/presentations/,,,
death_map_2007_vt.pdf

Hollins University Visitors
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Casebook: The Giant's Causeway

Casebook: The Giant's Causeway

On the northeast coast of Ireland, there is a "paved" area of 40,000 interlocking (mostly) hexagonal stone pillars, roughly the same size.

Some of the pillars reach up to a cliff, and form a staircase that disappears into the sea.

Could a natural process explain this?

Casebook: The Giant's Causeway

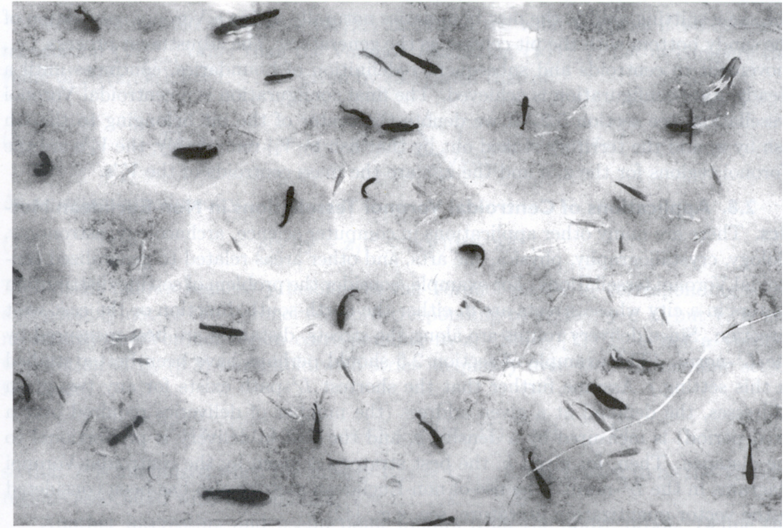
Casebook: The Spitting Fish

The *Tilapia mossambica* is sometimes called the spitting fish.

1. It raises its young in its mouth, spitting them out when they're ready.
2. It breeds by building a nest on the river bottom, picking up stones and spitting them away

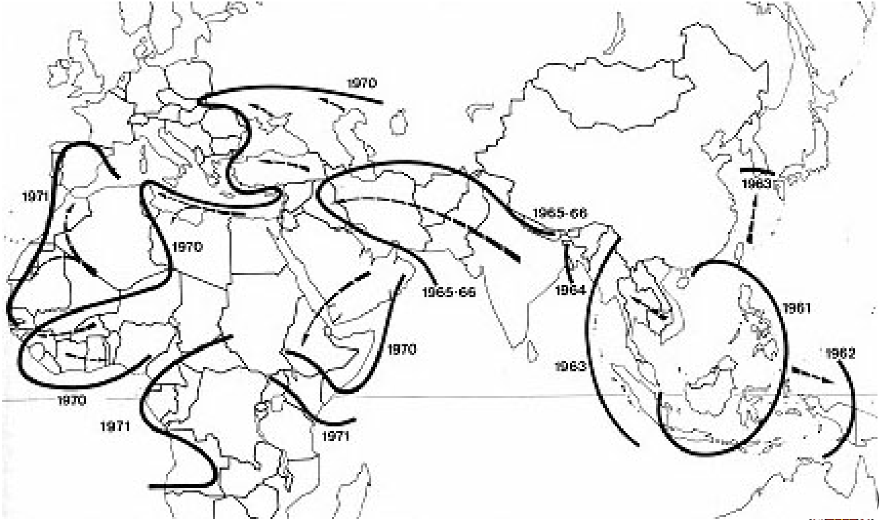
Without taking geometry class, these fish build polygonal networks of nests

Casebook: The Spitting Fish



- Each cell is about the same size;
- Each cell is about the same shape;
- The cells are “centered”.

As far as we know, the fish do not use a blueprint, nor do they have a planning committee meeting!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In the early nineteenth century, European doctors working in India reported a strange new disease called **cholera**.

Cholera was agonizing and fatal.

No one knew how it was transmitted. No one had any idea how to treat or prevent it.

Yearly reports showed that cholera had begun to move across Asia, and into southern Europe. Soon it showed up here and there in England.

This picture, illustrating a cholera epidemic, displays one common theory of cholera's transmission: **transmission by miasm**.

A miasm was a hypothetical airborne cloud, smelling terrible, and carrying the disease.

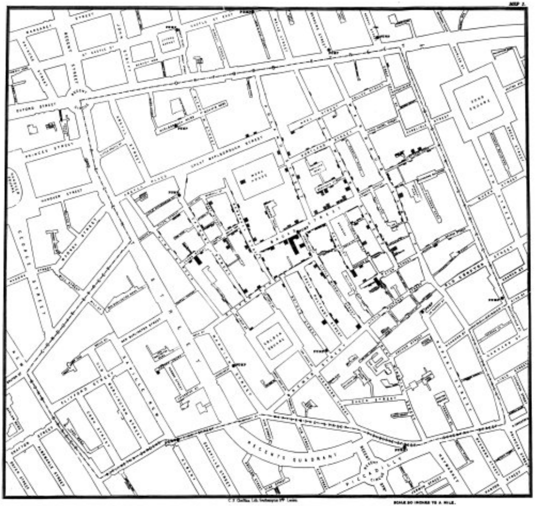
Miasm explained why cholera victims often lived in poor areas full of tanneries, butcher shops, and general dirty conditions.

It also explained why cholera did not simply spread out across the entire population, but seemed to travel, almost like the wind, settling down to devastate a neighborhood, and then moving on.

- 1: A Casebook
- **2: THE DEATHS IN GOLDEN SQUARE**
- 3: The Voronoi Diagram
- 4: Euler's Formula
- 5: Voronoi Computation
- 6: Centered Systems

Golden Square: Cholera Outbreak of 1854

Golden Square: A Map

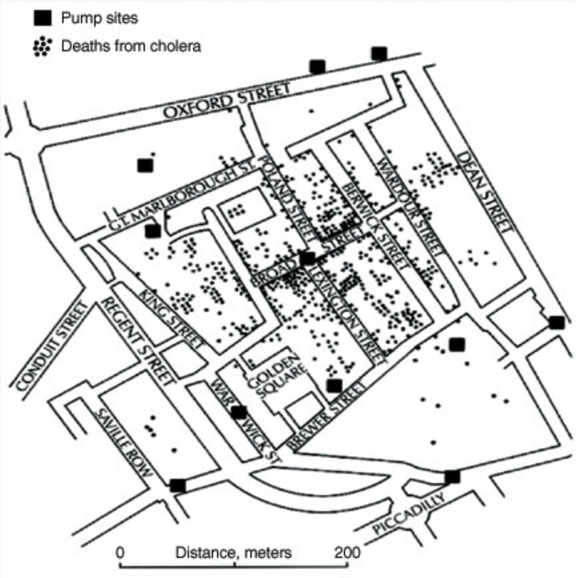


Golden Square: John Snow's Investigation

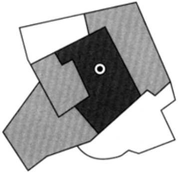
Dr John Snow suspected a particular water pump on Broad Street was the source of the cholera outbreak, but the pump water seemed relatively clean when he examined it.

- He made a map of where victims lived;
- He paced the distance to the nearest pump;
- He drew a line around the houses closest to the Broad Street pump.
- Some victims outside the line nonetheless used the pump.

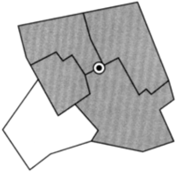
Golden Square: See the Pumps as "Suspects"



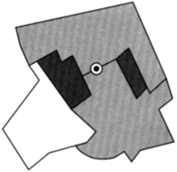
Golden Square: See the Pump Distance as Explanationl



In this aggregation of individual deaths into six areas, the greatest number is concentrated at the Broad Street pump.



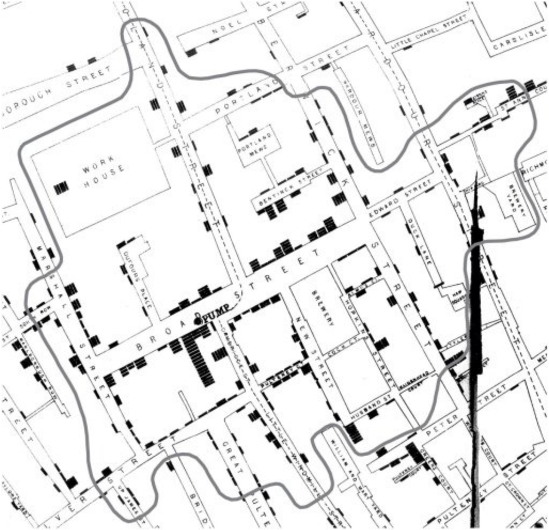
Using different geographic subdivisions, the cholera numbers are nearly the same in four of the five areas.



In this aggregation of the deaths, the two areas with the most deaths do not even include the infected pump!

[18] Mark Monmonier, *How to Lie with Maps* (Chicago, 1991), pp. 142–143.

Golden Square: The Pump and its Neighborhood



Golden Square: Conclusion

People minimize the distance they travel;

The path of a disease can be determined by local traffic patterns.

Understanding local geometry reveals natural neighborhoods and connections.

Voronoi

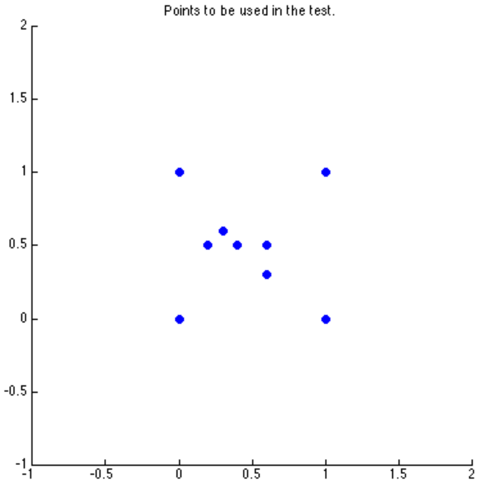
- 1: A Casebook
- 2: The Deaths in Golden Square
- **3: THE VORONOI DIAGRAM**
- 4: Euler's Formula
- 5: Voronoi Computation
- 6: Centered Systems

Voronoi: Names of Things

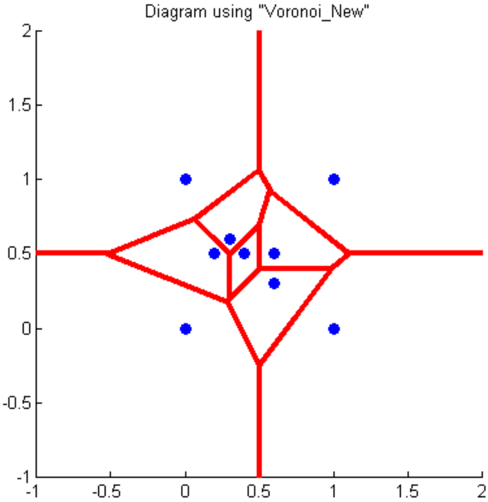
Each point is given all the area that is nearest to it.

- **generators**– the points;
- **regions**–the areas;
- **edges**–the boundary lines;
- **vertices**–where 3 lines meet;
- **Voronoi diagram**–this division of space

Voronoi: 9 points



Voronoi: Diagram for 9 points



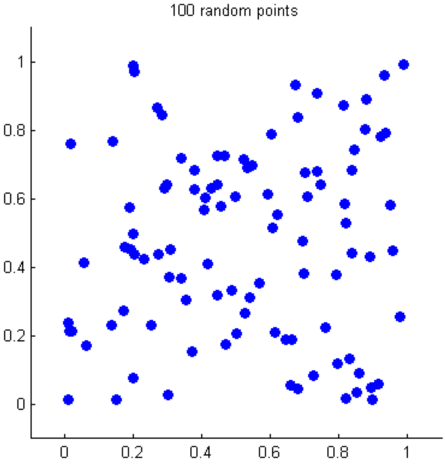
Voronoi: Properties of the Regions

- All regions are convex (no indentations);
- All regions are polygonal;
- Each region is the intersection of half planes;
- The infinite regions have generators on the convex hull;

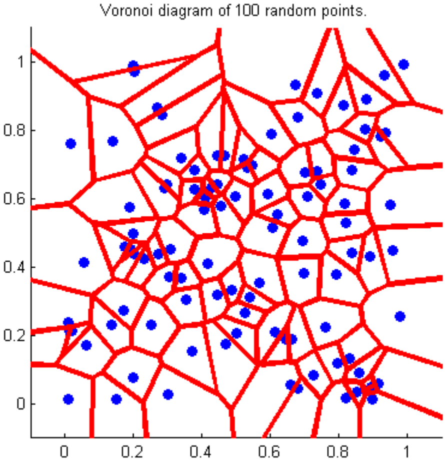
Voronoi: Properties of the Edges and Vertices

- Edges are perpendicular bisectors of neighbor lines;
- Vertices are equidistant from 3 generators;
- Each vertex is the center of a circle through 3 generators;
- Each vertex circle is empty (no other generators).

Voronoi: 100 points



Voronoi: Diagram of 100 points



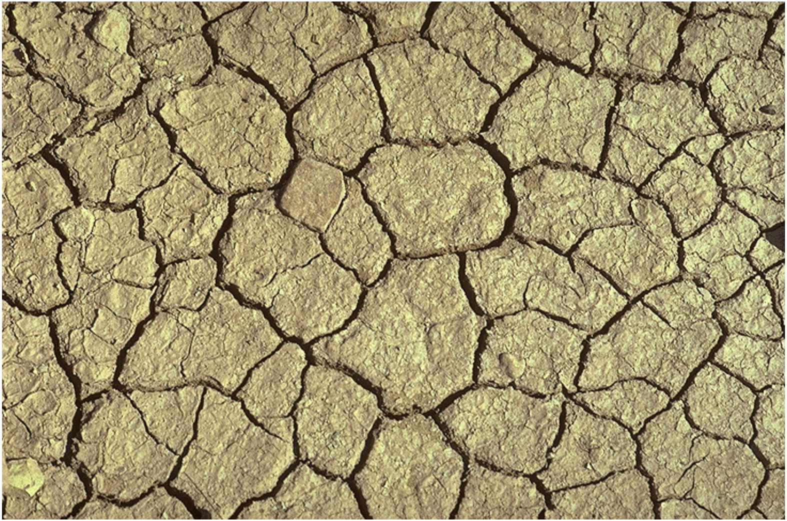
The Voronoi diagram seems to create some order out of scattered points.

We might compare the Giant's Causeway, or the spots on leopards or giraffes, or the "spitting fish"" nests.

However, the generators are not "centered" and the shape and size of the regions seems to vary more than in some natural patterns. Can we explain or control this?

Voronoi: Giraffe Patches

Voronoi: Drying Mud

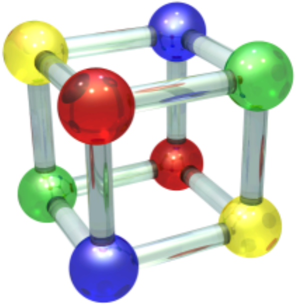


Leonhard Euler developed a formula that related the number of faces, edges and vertices in a 3-dimensional polyhedron.
This formula can help us estimate the "size" of a Voronoi diagram, that is, the number of edges and faces we might find for a given number of points.

Euler's Formula for 3D Polyhedrons

Euler's formula relates **F**aces, **V**ertices, and **E**dges.

$$F + V = E + 2$$



$$6 + 8 = 12 + 2$$

Euler's Formula: Adapted to 2D

We can apply Euler's Formula to a 2D figure in the plane

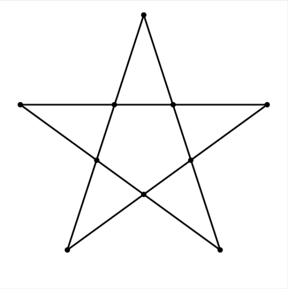
Just imagine the surface is stretchable. Puncture the surface at one point and flatten it out.

Our puncture will make one face disappear. Things will add up correctly in Euler's formula if we increase the number of faces we can see by 1, to account for the face we destroyed.

Euler's Formula for Bounded 2D Figures

In 2D, and a bounded figure, add one infinite face.

$$(F + 1) + V = E + 2$$



$$(6 + 1) + 10 = 15 + 2$$

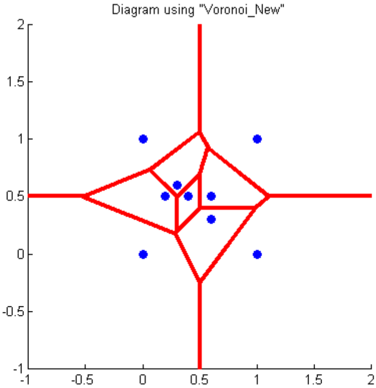
We can apply Euler's Formula to a Voronoi diagram.

The infinite regions are just "very big" faces.

Things will add up properly as long as we add one vertex at infinity.

In 2D, and unbounded diagram, add vertex at infinity.

$$\begin{aligned} F + (V+1) &= E + 2 \\ 9 + (11+1) &= 19 + 2 \end{aligned}$$



Each edge has 2 vertices. Each vertex belongs to (at least) 3 edges.

$$3V \leq 2E$$

Substituting for V in Euler's equation gives:

$$E \leq 3 * N - 6$$

Substituting for E in Euler's equation gives:

$$V \leq 2 * N - 4$$

So knowing N, we know limits for the sizes of E and V.
Each edge makes 2 neighbors, so (in the plane) our average number of neighbors must be slightly less than 6.

Voronoi Computation

- 1: A Casebook
- 2: The Deaths in Golden Square
- 3: The Voronoi Diagram
- 4: Euler's Formula
- **5: VORONOI COMPUTATION**
- 6: Centered Systems

Voronoi Computation

A Voronoi diagram seems like a “picture”.

How would you store it in a computer?

It’s almost a bunch of polygons, but of different shapes and sizes and orders - and some polygons are “infinite”.

At least Euler’s formula gives us limits on the number of edges and vertices.

Voronoi Computation: Algorithm 1

The Method of Half Planes

To compute the Voronoi region $R1$ for one generator $G1$:

Initially, assume the region $R1$ is the whole plane.

Take another generator $G2$. Only half the plane is closer to $G1$ than to $G2$. Intersect $R1$ with this half plane to get the new (smaller) $R1$.

Repeat for generators $G3$, ..., Gn, to determine $R1$.

To get every Voronoi region, must do this for each generator.

Byer's Method

For the generator $G1$, draw the "neighbor" lines to all other generators.

Draw the perpendicular bisector lines to each neighbor line, and "remember" where each intersection point is.

Starting with the intersection point closest to $G1$, the Voronoi region $R1$ is made up of segments of the closest bisector lines.

Fortune's method

Pass a horizontal "scanning" line from top to bottom.

Just in front of the scanning line, only a few of the generators are "active".

Behind the scanning line, the diagram is finished.

This method is extremely efficient; it doesn't waste time looking at generators that are far away from the area being scanned.

Voronoi Computation: Algorithm 4

Pixel method

Set up a graphics box of 500 x 500 pixels.

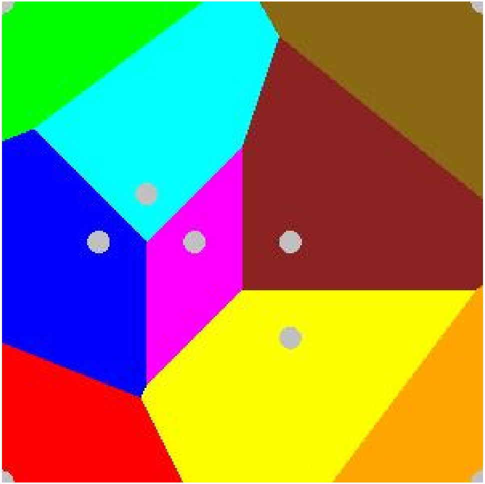
Pick a color to associate with each generator.

Plot the generators as pixels in the graphics box.

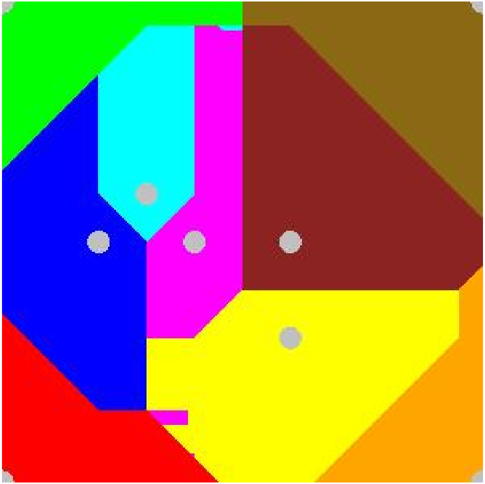
For each of the 250,000 pixels, determine the nearest generator pixel, and take its color.

This method is "stupid" but reliable. It's easy to adjust it to cases where the region has a strange shape, or distance is measured in another way.

Voronoi Computation: Pixel Method (Euclidean distance)



Voronoi Computation: Pixel Method (L1 distance)



Distance Contour method

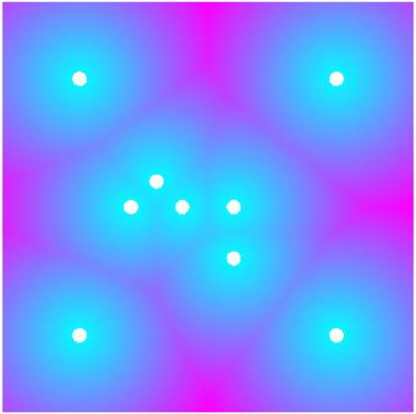
Set up a graphics box of 500 by 500 pixels.

Plot the generators as pixels in the graphics box.

For each of the 250,000 pixels, determine the distance to the nearest generator.

Use graphics software to create a contour plot of the distance function, which is 0 at the generator, and rises up like a paraboloid.

Voronoi Computation: Distance Contour Plot



Voronoi Computation: Distance Surface Plot

A side view, using a different “mountain chain”.

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- **6: CENTERED SYSTEMS**

Physical examples differ from our mathematics.

- Instead of the plane, we have a finite area.
- The regions are roughly the same size.
- The regions are roughly the same shape.

Centered Systems: Center the Generators

Suppose, once we compute a Voronoi region, we allow the generator to move towards the center.

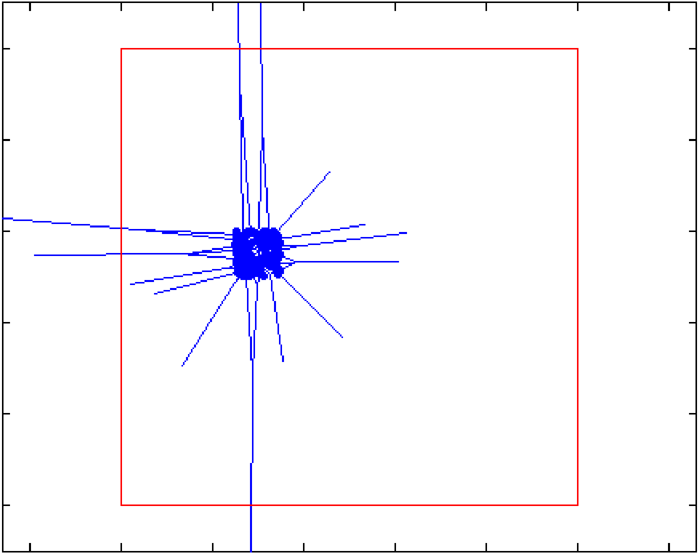
This is like relocating the capital city of a state to the center of the state (this happened in Iowa!)

One effect: probably reduces more distances to the generator.

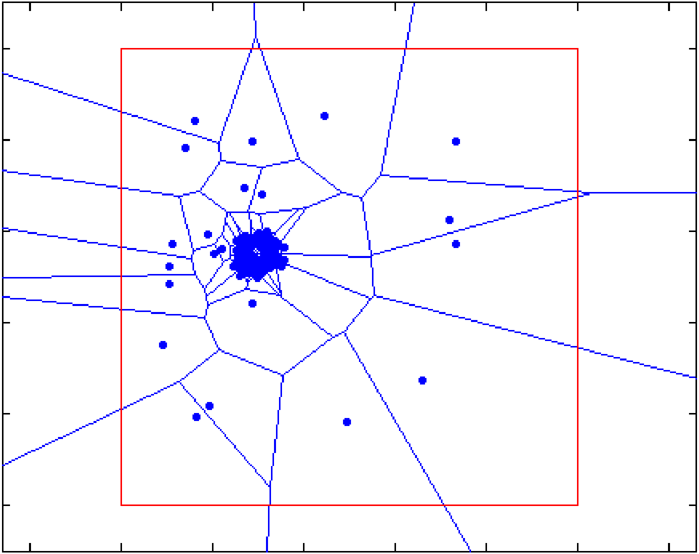
Another effect: some points may suddenly "belong" to a different generator.

So what happens if we recompute the Voronoi diagram?

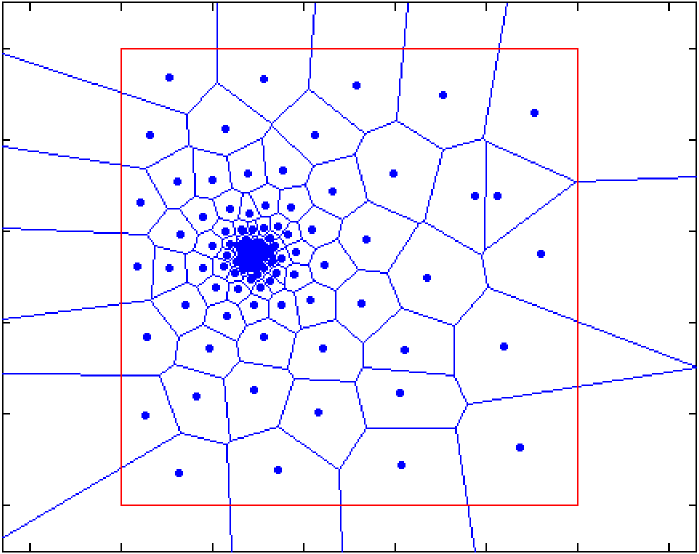
Centered Systems: Step 001



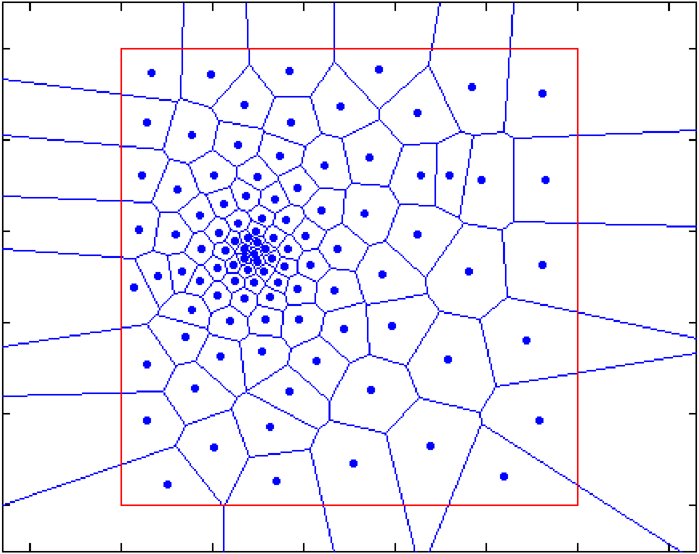
Centered Systems: Step 002



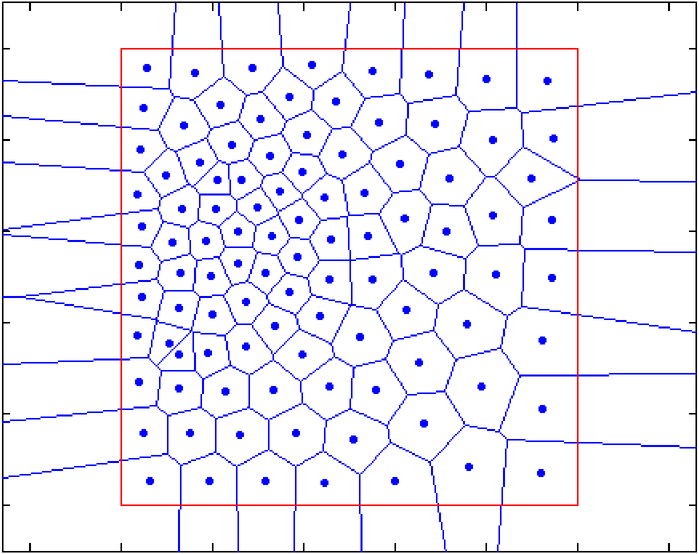
Centered Systems: Step 010



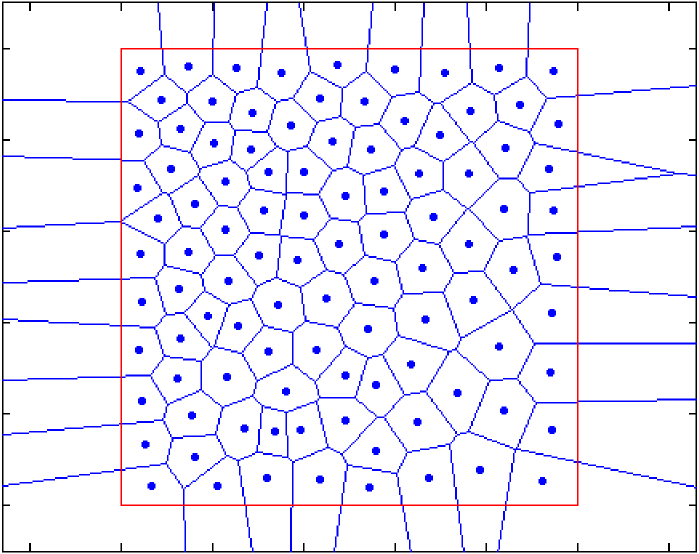
Centered Systems: Step 020



Centered Systems: Step 040



Centered Systems: Step 080



Conclusion: Voronoi Diagrams

The Voronoi diagram is a mathematical tool for analyzing geometric processes with many local centers.

- disease outbreaks
- patterns on animal skin or sea shells
- territories of ant colonies, prairie dogs
- how can a plane best avoid all enemy bases?

Conclusion; Centralized Voronoi Diagrams

The Centroidal Voronoi diagram allows us to understand the existence of centers, or to search for the best arrangement of them:

- how do fish adjust their nesting sites?
- how do rotating cells form in a cooling liquid?
- placement of mailboxes
- arrangement of sonar receivers on ocean floor
- what arrangement of bases makes it hardest for planes to penetrate?

Conclusion: The John Snow Memorial