PDE Model Reduction Using SVD

https://people.sc.fsu.edu/∼jburkardt/presentations/...
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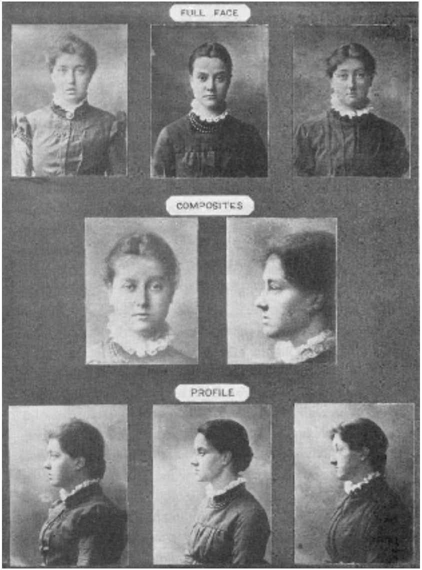
Example: A 9-D Space with a 2-D Subspace (Sirovich)

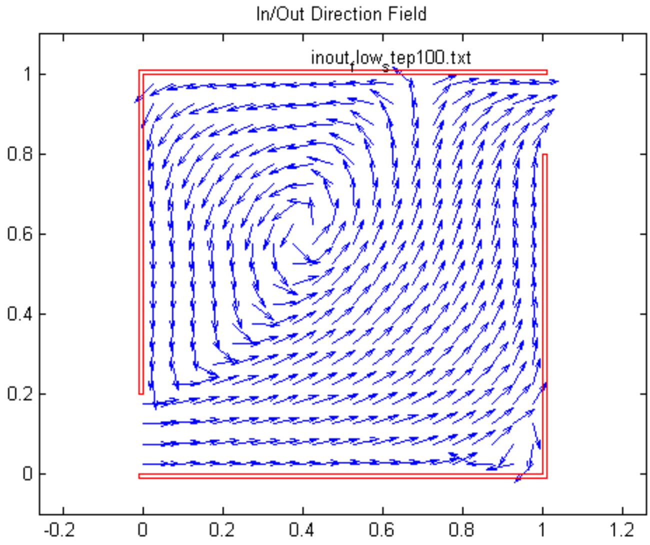
Example: Composite Photographs (Francis Galton)



Example: Facial Recognition (Muller, Magaia, Herbst)

Image from Muller, Magaia, Herbst

Example: Fluid Flow with Varying Input



Model Reduction

Earth satellites generate too much data to send; most of what is received is sent to archives and ignored.

Particle accelerators record billions of collisions; human investigators only see the tiny fraction selected by computer scanning.

Cheap data floods the world, but the information still hides inside.

To extract that information: *model reduction* or *reduced order modeling*..

Model Reduction

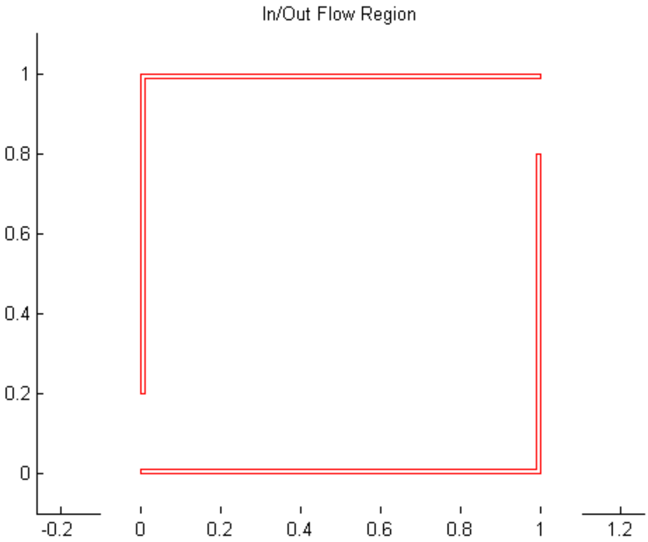
Suppose we have N records, each of M computer words.

Suppose there are patterns and structures in this data.

We want to keep most of the information, but using less than $M * N$ computer words.

Also, can we detect and exhibit the structures?

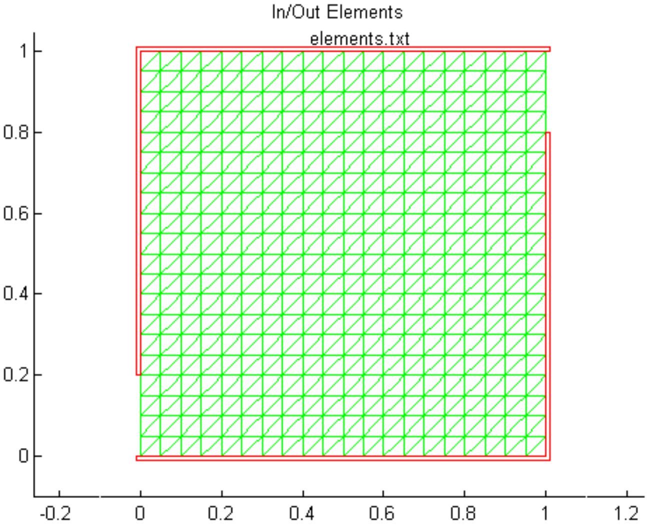
The IN/OUT Flow: region



Navier-Stokes equations for velocity vector $u(x, y)$ and pressure $p(x, y)$:

$$\begin{aligned} \rho \mathrm{u}_t - \mu \Delta \mathrm{u} + \rho(\mathrm{u} \cdot \nabla)\mathrm{u} + \nabla p &= 0 \\ \rho \nabla \cdot \mathrm{u} &= 0 \end{aligned}$$

We create a mesh of “elements”.



Finite Elements: Function approximation

FEM approximates velocity and pressure using basis functions generated by mesh:

$$
\begin{aligned}
u(x,y) &\approx U(x,y) = \sum_{i=1}^{NV} a_i \quad \phi_i(x,y) \\
p(x,y) &\approx P(x,y) = \sum_{j=1}^{NP} b_j \quad \psi_i(x,y)
\end{aligned}
$$

Pack coefficients a and b into one vector c.

The set of possible vectors C are now a "model" of the problem. (but certainly a "fat" model, in need of reduction.)

Finite Elements: State equations

For true solutions u and p of the state equations, the momentum and continuity equations are satisfied exactly, and everywhere in the region.

$$\begin{aligned} Momentum(u(x,y),p(x,y)) &= 0 \\ Continuity(u(x,y),p(x,y)) &= 0 \end{aligned}$$

Finite Elements: Galerkin method

- form approximate $U(x,y)$, $P(x,y)$;
- evaluate state equation residuals for U and P.
- require orthogonality of residuals to corresponding basis functions.

$$
\begin{aligned}
\int_\Omega Momentum(U(x,y),P(x,y)) \quad \phi_i(x,y)d\Omega &= 0 \\
\int_\Omega Continuity(U(x,y),P(x,y)) \quad \psi_j(x,y)d\Omega &= 0
\end{aligned}
\qquad (1)
$$

The IN/OUT Flow: Solution

- 41 by 41 grid of nodes;
- 800 quadratic triangular elements;
- $M = 3{,}362$ velocity coefficients.
- Solve for N=500 time steps.

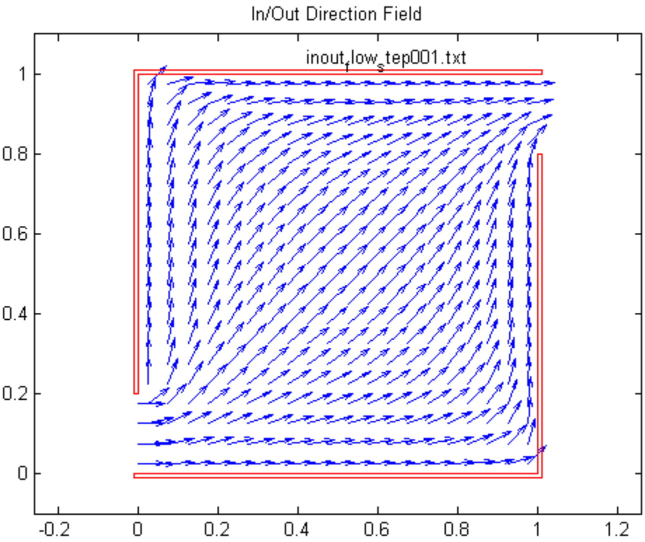
The IN/OUT Flow, Boundary Conditions

We control the strength of the parabolic inflow at lower left.

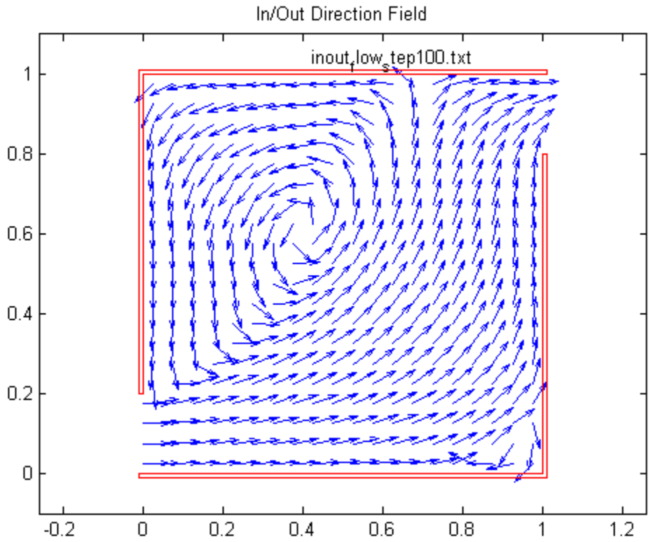
Inflow strength abruptly changed at step 250.

During startup and for boundary shock at step 250 the flow will exhibit a wide range of transient patterns.

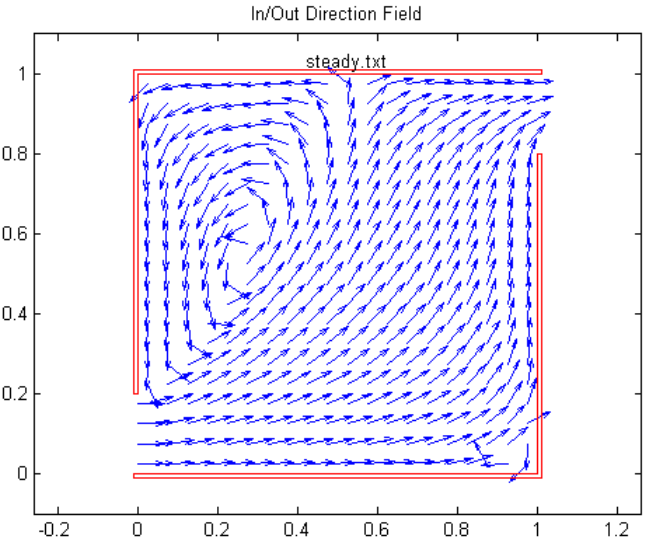
The IN/OUT Flow, Direction Field, Time Step 1



The IN/OUT Flow, Direction Field, Time Step 100



The IN/OUT Flow, Direction Field, Steady State



FACES: A Second Set of Data

200 people were each photographed in 3 poses for N=600 images.

If each image used 400 * 800 pixels, this makes about 300,000 pixels, each with R, G and B values, for about M=1,000,000 numeric values.

SVD: The Singular Value Decomposition

The singular value decomposition will be our key to reduced order models.

The SVD of an M by N (real) matrix A:

$$A = U \cdot \Sigma \cdot V'$$

- U is M by M orthogonal;
- Σ is an M by N (nonnegative) diagonal matrix.
- V is N by N orthogonal;

SVD: Diagram

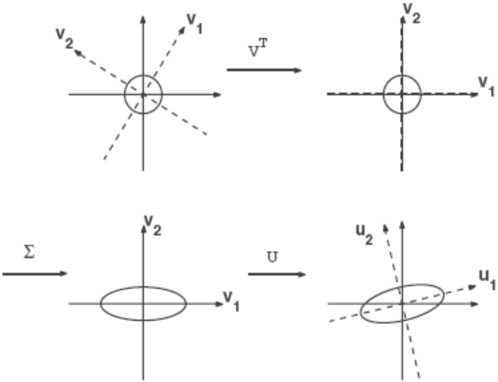


Image from Muller, Magaia, Herbst

SVD: What the SVD Tells Us

For a system with patterns, the SVD decomposition tells us where the information is concentrated.

$$A = U \cdot \Sigma \cdot V'$$

- The leading columns of U are the "preferred behaviors";
- The diagonal of Σ is an energy or importance weight;
- The entries of Σ are positive, and sorted in decreasing order;
- For information with patterns, the entries of Σ are rapidly decreasing.

SVD: Creating a Reduced Model

The reduced model chooses the first L columns of U. (L might be prespecified, or determined by Σ)

Replace Σ by $\widehat{\Sigma}$, zeroing entries beyond L.

The "interesting" or dominant information in A should be well described by

$$\widehat{A} = U \cdot \widehat{\Sigma} \cdot V'$$

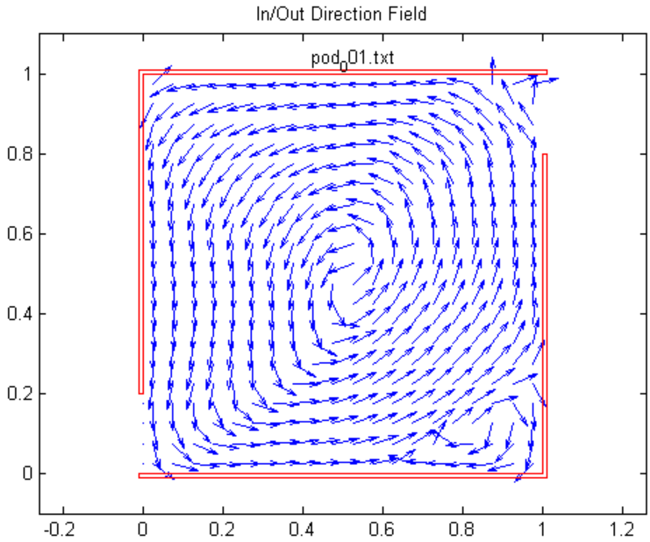
SVD Model Reduction for IN/OUT

500 flow fields were computed.

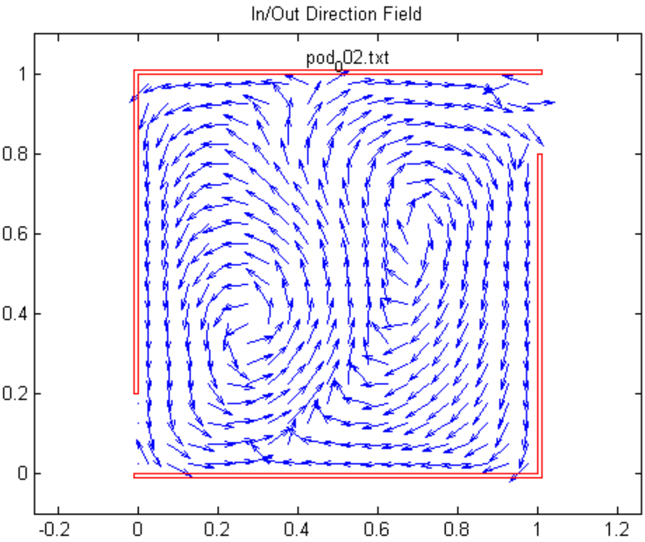
Subtracting a multiple of the steady state solution "normalized" the data to **zero boundary conditions**.

The 16 dominant (and orthogonal) modes were extracted.

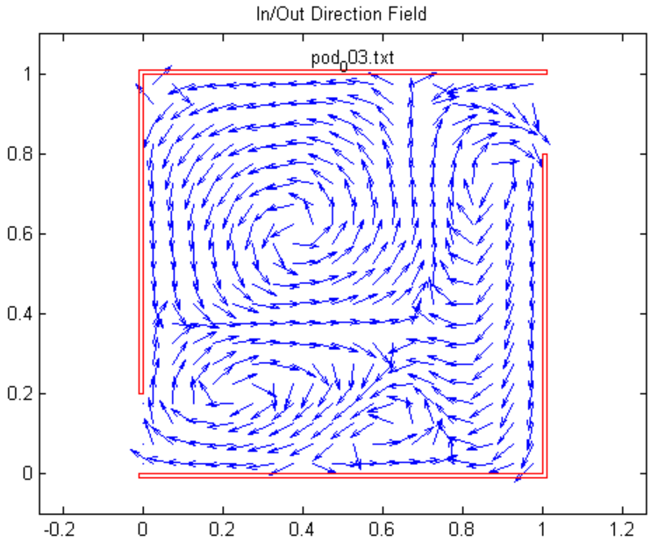
SVD Model Reduction for IN/OUT: Vector 1



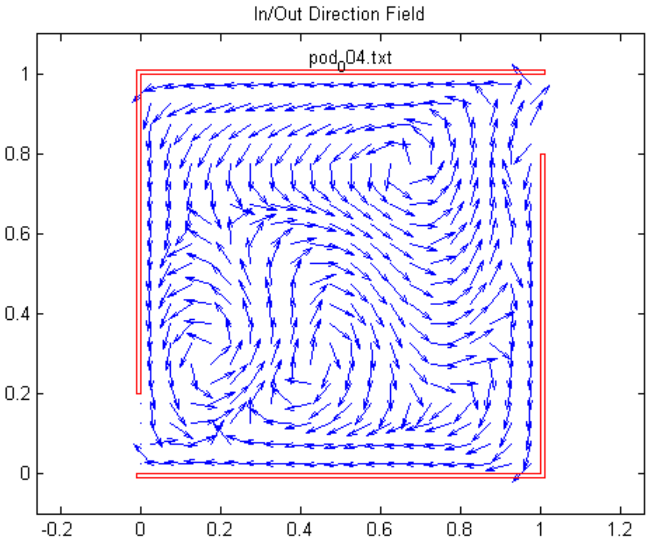
SVD Model Reduction for IN/OUT: Vector 2



SVD Model Reduction for IN/OUT: Vector 3



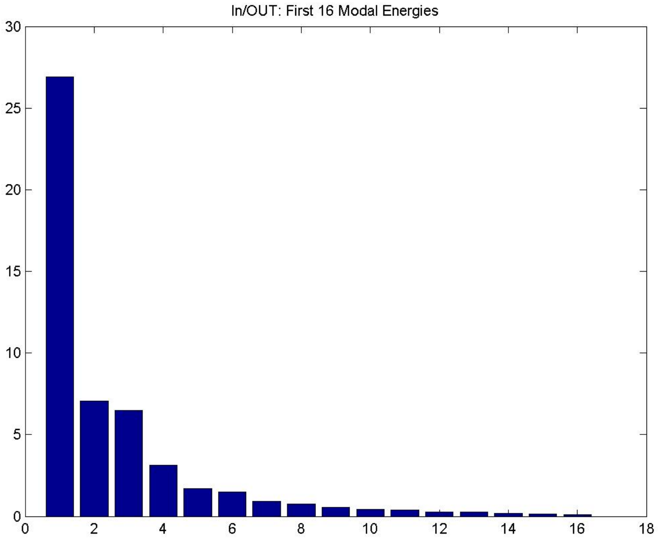
SVD Model Reduction for IN/OUT: Vector 4



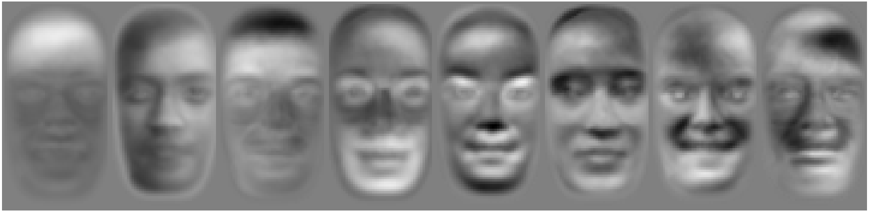
SVD Model Reduction for IN/OUT: Vector "Importance"

Index	Value	Relative	Cumulative
1	26.9107	0.527	0.527
2	7.0878	0.138	0.666
3	6.5015	0.127	0.794
4	3.1420	0.061	0.855
5	1.6973	0.033	0.889
6	1.4947	0.029	0.918
7	0.9253	0.018	0.936
8	0.7592	0.014	0.951
9	0.5738	0.011	0.962
10	0.4570	0.008	0.971
...	...	...	...
16	0.0994	0.001	0.997

SVD Model Reduction for IN/OUT: Vector "Importance"



SVD Model Reduction for FACES



Using similar SVD methods on the face data, here are first 8 modes or "eigenfaces".

Usually 100 or 150 "eigenfaces" are enough to store almost all the facial information.

Image from Muller, Magaia, Herbst

SVD Model Reduction for FACES: Vector "Importance"

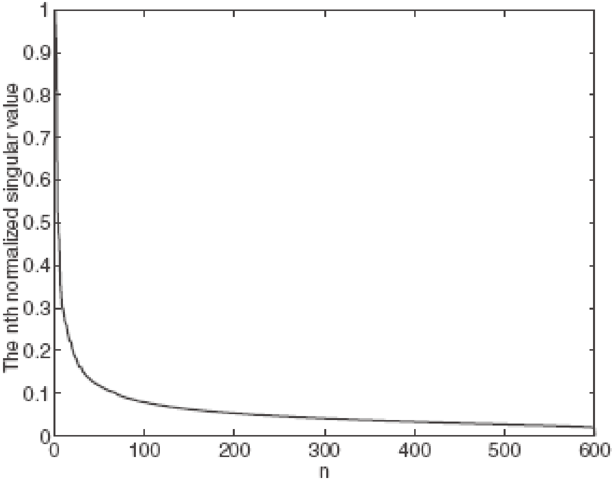


Image from Muller, Magaia, Herbst

SVD Model Reduction: An Empirical Model

For the IN/OUT flow, we started with a model, namely the Navier Stokes equations.

For faces, we have no prior "theory" or formulas for what faces should do.

In both cases, the SVD identified the important basis functions.

The SVD can form **an empirical model** without any underlying theory.

ROM: Reduced Order Modeling

Reduced order modeling applied to the data extracts an orthogonal basis, in order of importance.

The singular values tell us how much information we are saving from the raw data.

If the raw data is a good sample, and the SVD allowed us to select a small but very important basis, then we may accidentally have an empirical basis set, a sort of accidental physical law that we can use.

ROM: Representing New Data

"Eigenfaces" extracted and compressed information from face dataset.

Can the eigenface basis set recognize new faces?

The flow vectors represented dominant modes in the flow field.

Can we solve new problems in the same region using this smaller basis?

ROM: Solving New IN/OUT Problems

Any flow vector (original data, or extracted singular vector)

- is a sum of the finite element basis vectors
- defines a flow function $(U(x,y), P(x,y))$.

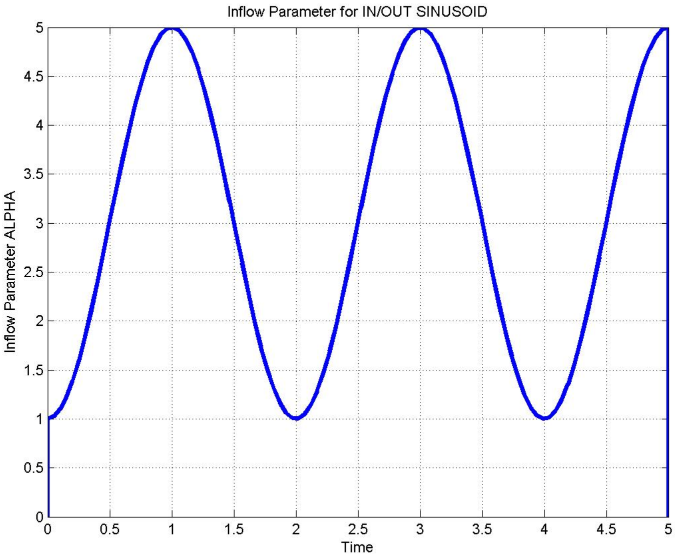
The extracted singular vectors:

- may or may not satisfy boundary conditions;
- satisfy the continuity equation;

Linear combinations of the first few singular vectors can represent most of the behavior of our data.

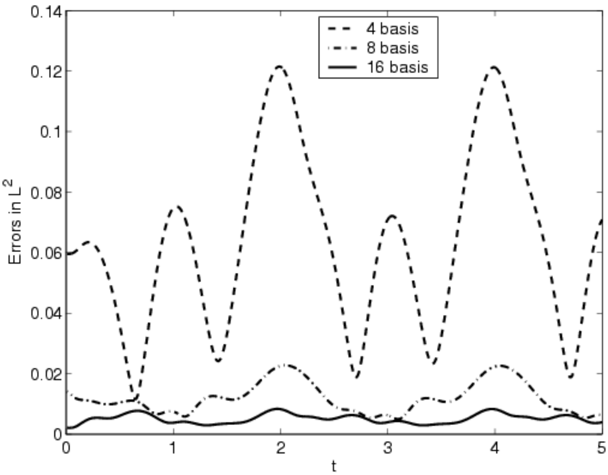
Why not use them as a new finite element basis?

ROM: Solving New IN/OUT Problems



The strength of the boundary input flow varies in new ways.

ROM: Solving New IN/OUT Problems



L2 difference between ROM and full FEM solutions.

Reconstruction using 40, 100, 450 eigenfaces.

Image from Muller, Magaia, Herbst

ROM: The Covariance Matrix

The SVD includes information that can be used to recognize or reject new data. If we "gently" preprocess the data, we get usable covariance information.

- Subtract average from all data;
- Scale, dividing by $\sqrt{N}$;
- U vectors are maximum variation directions
- Σ contains standard deviations

ROM: The Covariance Matrix

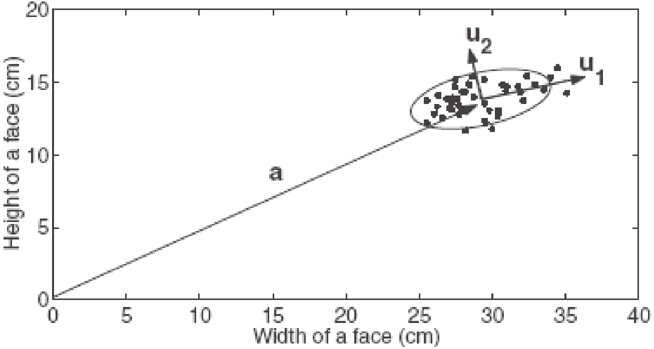


Image from Muller, Magaia, Herbst

ROM: The Covariance Matrix

Suppose we have a new item. Is it a face?

Subtract the average, and project it onto the U vectors.

From the SVD information, we can compute the probability that an item of data would fall this far from the average.

Summary of ROM

- SVD is a key approach.
- unknown information and patterns discovered.
- compact representation of huge dataset;
- a natural (secondary) basis for FEM calculations.
- projection can be used to “recognize” new data.

References

Burkardt, Gunzburger, Lee, *Centroidal Voronoi Tessellation-Based Reduced-Order Modeling of Complex Systems*, SIAM Journal on Scientific Computing, Volume 28, Number 2, 2006.

Muller, Magaia, Herbst, *Singular Value Decomposition, Eigenfaces, and 3D Reconstructions*, SIAM Review, Volume 46, Number 3, 2004.