Example for stationary points

Find all stationary points of the function:

$$f(x) = 2x_1^3 - 3x_1^2 - 6x_1x_2(x_1 - x_2 - 1) \qquad x = (x_1, x_2)^T$$

and show which points are minima, maxima or neither.

Solution:

Find stationary points:

$$\frac{\partial f}{\partial x_1} = 6(x_1 - x_2)(x_1 - x_2 - 1)$$
$$\frac{\partial f}{\partial x_2} = 6x_1(-x_1 + 2x_2 + 1)$$

Stationary points are solutions of the system: $\begin{cases} (x_1 - x_2)(x_1 - x_2 - 1) = 0 \\ x_1(-x_1 + 2x_2 + 1) = 0 \end{cases}$

There are four possible cases:

1. $\begin{cases} x_1 - x_2 = 0 \\ x_1 = 0 \end{cases} \Rightarrow x_1 = 0, x_2 = 0$

The first point is A (0,0)

2. $\begin{cases} x_1 - x_2 = 0 \\ -x_1 + 2x_2 + 1 = 0 \end{cases} \Rightarrow x_1 = x_2, x_1 + 1 = 0 \Rightarrow x_1 = x_2, x_1 = -1 \Rightarrow x_1 = -1, x_2 = -1$

The second point is B (-1, -1)

3. $\begin{cases} x_1 - x_2 - 1 = 0 \\ x_1 = 0 \end{cases} \Rightarrow \begin{cases} x_2 = -1 \\ x_1 = 0 \end{cases}$

The third point is C (0,1)

4. $\begin{cases} x_1 - x_2 - 1 = 0 \\ -x_1 - 2x_2 + 1 = 0 \end{cases} \Rightarrow \begin{cases} x_2 = 0 \\ x_1 = 1 \end{cases}$

The fourth point is D (1,0)

We have 4 stationary points namely A, B, C and D.

Step 2

<u>The nature of the stationary points</u>

Compute the Hessian matrix:

$$\begin{bmatrix} \dfrac{\partial^2 f}{\partial x_1^2} & \dfrac{\partial^2 f}{\partial x_1 \partial x_2} \\ \dfrac{\partial^2 f}{\partial x_2 \partial x_1} & \dfrac{\partial^2 f}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} 12x_1 - 12x_2 - 6 & -12x_1 + 12x_2 + 6 \\ -12x_1 + 12x_2 + 6 & 12x_1 \end{bmatrix}$$

Now we can determine the nature of each stationary points by considering value of Hessian at that point and whether Hessian is positive definite, negative definite or indefinite.

1. **A (0,0)**

$$\|G(0,0| = \begin{vmatrix} -6 & 6 \\ 6 & 0 \end{vmatrix} = -36 < 0$$

A (0,0) is not a minimum and not a maximum since G at that point is indefinite.

2. **B (-1, -1)**

$$|G(-1,-1)| = \begin{vmatrix} -6 & 6 \\ 6 & -12 \end{vmatrix} = 72 - 36 = 36 > 0$$

B (-1, -1) is a maximum since we can verify that for all x, $x^T Ax < 0$ i.e. negative definite.

3. **C (0, -1)**

$$|G(0,-1)| = \begin{vmatrix} 6 & -6 \\ -6 & 0 \end{vmatrix} = -36 < 0$$

C (0, -1) is not a minimum nor a maximum since it does not satisfy that for all x, $x^T Ax < 0$ or $x^T Ax > 0$. The eigenvalues of G at this point have both positive and negative entries (verified by Gaussian elimination)

4. **D (1,0)**

$$|G(1,0)| = \begin{vmatrix} 6 & -6 \\ -6 & 12 \end{vmatrix} = 36 > 0$$

D (1,0) is a minimum since G is positive definite i.e. for all x i.e. $x^T Ax > 0$.

Solution

(-1, -1) is a maximum

(1,0) is a minimum