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Spectral stochastic two-scale convergence method for parabolic PDEs

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SUMMARY

Following the theory of two-scale convergence method introduced by Nguetseng (*SIAM J. Math. Anal.* 1989; **20**:608–623) and further developed by Allaire (*SIAM J. Math. Anal.* 1992; **23**:1482–1518), we introduce the chaos two-scale method as a spectral stochastic tool to tackle parabolic partial differential equations where the material properties are stochastic processes $\sigma^{\varepsilon}(t,x,\omega)$ of the form $\sigma(t,x,t/\varepsilon^{\gamma},x/\varepsilon,\omega)$, oscillating in both space and time variables with different speeds. Periodicity with respect to the fast or local variables is assumed, and, stationary Gaussian material properties processes are considered. Copyright © 2010 John Wiley & Sons, Ltd.

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KEY WORDS: two-scale convergence method; periodic homogenization; Karhunen–Loève expansions; Wiener polynomial chaos; spectral methods

NOTATIONS

The following notations are frequently used:
$\mathcal{O}$: designates an open set of $\mathbb{R}$
$Y=]0,l[$: denotes the unit cell also referred to as the reference period, here l designates the period.
$\mathbb{S}^2=]0,T[\times\mathcal{O}$
$\mathbb{S}^2_{\#}=[0,1]\times Y$
$\mathbb{S}^3=]0,T[\times\mathcal{O}\times Y$
$\mathbb{S}^4=]0,T[\times\mathcal{O}\times[0,1]\times Y$
$\mathrm{d}\mu=\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}\tau\,\mathrm{d}y$: the Lebesgue measure over $\mathbb{S}^4$
$\mathscr{C}^k(\mathcal{O})$: the space consisting of all functions f which, together with all their partial derivatives $\delta^{\alpha}f$ of orders $|\alpha|<k$, are continuous on $\mathcal{O}$.
$\mathscr{C}^k_{\#}(Y)$: the space of functions $f\in\mathscr{C}^k(Y)$ and Y-periodic.
$\mathbb{L}^2(\mathcal{O})$: the space of measurable functions $f:\mathcal{O}\to\mathbb{R}$ for which $\{\int_{\mathcal{O}} f^2\,\mathrm{d}x\}^2<\infty$
$\mathbb{L}^2_{\#}(Y)$: the space of measurable functions $f\in\mathbb{L}^2(Y)$ and Y-periodic
$\mathbb{H}^1(\mathcal{O})$: the space consisting of all integrable functions $f:\mathcal{O}\to\mathbb{R}$ whose first-order weak derivatives exist and are square integrable, $\mathbb{H}^1(\mathcal{O})=\{f\,|\,f,\nabla f\in\mathbb{L}^2(\mathcal{O})\}$
$\mathbb{H}^1_{\#}(\mathcal{O})=\{f\in\mathbb{H}^1(Y),\ f \text{ is } Y\text{-periodic}\}$

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$\mathbb{L}^2(]0,T[;\mathbb{H}^1(\mathscr{O}))$: the space of functions that are square-integrable with respect to time and have square-integrable derivatives with respect to space:

$$\mathbb{L}^2(]0,T[;\mathbb{H}^1(\mathscr{O}))=\{f\,|\,f:]0,T[\rightarrow\mathbb{H}^1(\mathscr{O}),\int_0^T\|f(t,.)\|^2_{\mathbb{H}^1(\mathscr{O})}\,\mathrm{d}t<\infty$$

1. INTRODUCTION

Many problems of fundamental and practical importance exhibit multiscale phenomena, so that the task of computing or even representing all scales is computationally very expensive unless the multiscale nature of the problem is exploited in a fundamental way. Some examples of practical interest include, continuum mechanics of inhomogeneous media, composites, polycrystals and smart materials, fluid flow in porous media and turbulent transport in high Reynolds number flows, the deformation of saturated porous medium, the sound propagation through a liquid populated sparsely by bubbles, linear and non-linear wave propagation problems involving slow modulation of near periodic waves, terabyte data mining, as well as image processing display behaviors at different scales.

A detailed analysis of these problems at the smallest relevant scale, while conceptually possible, is rather prohibitive. For example, in the analysis of turbulent transport problems, the convective velocity field fluctuates randomly and contains many scales depending on the Reynolds number of the flow. In composite materials, the dispersed phases, which may be randomly distributed in the matrix, give rise to fluctuations in the thermal or electrical conductivity; moreover, the conductivity is usually discontinuous across the phase boundaries. The main difficulty in practical computations is often the presence of very different scales in the problem. On a grid that must cover the domain of the independent variables, it may be impossible to resolve highly oscillatory components well in the solution. A natural question is whether some averaged quantities of the solutions can still be accurately computed.

A useful and effective approach to the above-mentioned problems has been proposed involving the notion of homogenization of partial differential equations. One can refer to the pioneering work of Babuška [1, 2], Bensoussan *et al.* [3], and Sanchez-Palenchia [4]. More recently, an increasing number of books have appeared on the subject, Jikov *et al.* [5], Cioranescu and Donato [6], Pavliotis and Stuart [7], and Efendiev and Hou [8] to cite but a few.

Roughly speaking, homogenization is a rigorous adaptation, of what is known in physics or mechanics as averaging, to partial differential equations; it extracts homogeneous effective parameters from models of disordered or heterogeneous media through convergence analysis applied to the equations. Various concepts of convergence, such as G-convergence, Γ-convergence have been developed for this purpose.

When the homogenized problem has a non-local structure coupling between micro- and macrostructures or when the coefficients are of the form $\sigma(x,\frac{x}{\varepsilon})$, the usual homogenization techniques are somewhat difficult to apply and more elaborate forms of the multiple scale expansions are needed as described in Bensoussan *et al.* [3].

The method of two-scale convergence is a powerful one for studying homogenization problems for partial differential equations with periodically oscillating coefficients. The method was devised by Nguetseng [9] and further improved by Allaire [10, 11] and E [12, 13]. It was used to study problems of fluid flow through porous formations in [14]. The method was applied to transport equations with incompressible velocity field in [12] and [15]. The two-scale convergence method was extended to the case of non-periodic oscillations by Mascarenhas and Toader [16]. The concept of stochastic two-scale convergence in the mean has been introduced in Bourgeat *et al.* [17]. A striking advantage of the two-scale convergence method is that the homogenized and local problems appear directly as convergence results and do not have to be derived by tedious and somewhat dubious calculations. In practice, multiplying the global equation, also known as ε equation, by a test function of the type $\phi(x,\frac{x}{\varepsilon})$ and applying theorems yields both the local and the homogenized equations, and the proof of the convergence.

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In this paper, we study the spectral stochastic homogenization of the parabolic partial differential equation in the presence of random force and/or when the oscillating coefficient representing the material property is random.

$$\begin{cases} \dfrac{du^{\varepsilon}}{\mathrm{d}t}(t,x,\omega) - \dfrac{\partial}{\partial x}\left\{\sigma^{\varepsilon}(t,x,\omega)\dfrac{\partial u^{\varepsilon}}{\partial x}(t,x,\omega)\right\} = f(t,x,\omega) \quad \forall (t,x,\omega) \in]0,T[\times\mathcal{O}\times\Omega, \\ u^{\varepsilon}(t,x,\omega) = 0 \quad \forall (t,x,\omega)\in]0,T[\times\partial\mathcal{O}\times\Omega, \\ u^{\varepsilon}(t=0,x,\omega) = a(x) \quad \forall (x,\omega)\in\mathcal{O}\times\Omega, \end{cases}$$

where $\sigma^{\varepsilon}(t,x,\omega)$ are now stationary Gaussian stochastic processes oscillating in both time and space with dissimilar speed of the form

$$\sigma^{\varepsilon}(t,x,\omega) = \sigma\left(\frac{t}{\varepsilon^{\gamma}}, \frac{x}{\varepsilon}, \omega\right) \quad \text{or} \quad \sigma^{\varepsilon}(t,x,\omega) = \sigma\left(t, \frac{t}{\varepsilon^{\gamma}}, x, \frac{x}{\varepsilon}, \omega\right),$$

for γ, any positive real number and ε, a positive real number with $\varepsilon \searrow 0$. In this work, neither ε nor γ is assumed random. Consequently, no randomness in the fast variables is considered.

We are interested in the behavior and the numerical computation of the stochastic process $u^{\varepsilon}(t,x,\omega)$ as $\varepsilon \searrow 0$. In the absence of ω, the above problem is known as the heat equation, since it models the heat transfer in composite materials when the temperature u^{ε} is time-dependent. The deterministic problem is a particular case of the large class of parabolic partial differential equations. For homogenization results concerning the heat equation, we refer to Sanchez-Palenchia [4], Bensoussan *et al.* [3], Jikov *et al.* [5], and Cioranescu and Donato [6].

The contribution of this paper consists of two different aspects, an extension and an application of the two-scale convergence method to the spectral stochastic homogenization. We exploit the property of separation of deterministic variables from the random ones offered by the spectral representation of a stochastic process, we introduce the chaos two-scale convergence. We employ the results of Nguetseng [9] and Allaire [11] in the framework of spectral stochastic formulation. Then, an application to the aforementioned problems is conducted. The originality of the present work lies then in incorporating polynomial chaos to account for randomness, which was until recently handled by the Monte Carlo procedure. Another novelty resides in the numerical treatment, even at the deterministic level only, of parabolic PDEs where the material properties oscillate in both space and time variables with different speeds.

The paper is organized as follows: In Section 2, we use the results on the two-scale convergence method to derive local and global deterministic governing equations essential to the homogenization process. Section 3 is devoted to the spectral stochastic formulation. We define first the notion of the chaos two-scale convergence method and extend the results of Nguetseng [9] and Allaire [11] in a very natural way consisting in projecting the stochastic processes over an orthonormal basis known as the Wiener-chaos polynomials. Spectral stochastic formulation for both stochastic forcing process and stochastic material property process cases are then provided. The numerical procedure used in the paper is detailed in Section 4. The numerical results are presented and discussed in the same section. Section 5 is reserved for the summary and the concluding remarks.

2. DETERMINISTIC GOVERNING EQUATIONS

In this section we give the governing equations, their derivation relies heavily on the two-scale convergence method. The details can be found in the Appendix. We consider the second-order parabolic partial differential equation

$$
\mathscr{P}b=\begin{cases}\text{Find } u^{\varepsilon}(t,x)\\[2mm] \dfrac{\mathrm{d}u^{\varepsilon}}{\mathrm{d}t}(t,x)-\dfrac{\partial}{\partial x}\left\{\sigma^{\varepsilon}(t,x)\dfrac{\partial u^{\varepsilon}}{\partial x}(t,x)\right\}=f(t,x) \quad \forall (t,x)\in]0,T[\times\mathcal{O}\\[2mm] u^{\varepsilon}(t,x)=0 \quad \forall (t,x)\in]0,T[\times\partial\mathcal{O}\\[2mm] u^{\varepsilon}(t=0,x)=a(x) \quad \forall x\in\mathcal{O}\end{cases}
$$

where the function σ^{ε} is either of the form

$$
\sigma^{\varepsilon}(t,x)=\sigma\left(\frac{t}{\varepsilon^{\gamma}},\frac{x}{\varepsilon}\right) \tag{1}
$$

or

$$
\sigma^{\varepsilon}(t,x)=\sigma\left(t,\frac{t}{\varepsilon^{\gamma}},x,\frac{x}{\varepsilon}\right) \tag{2}
$$

where γ is a positive real number, ε is also a positive real number with $\varepsilon\searrow 0$.

Under the following assumptions

- For $T>0$, $\sigma^{\varepsilon}(t,x)\in\mathbb{L}^{\infty}(]0,T[\times\mathcal{O})$ and $\exists\beta>0$ such that: $\sigma^{\varepsilon}(t,x)\geqslant\beta$ $\forall t\in]0,T[$ and $\forall x\in\mathcal{O}$.
- $f\in\mathbb{L}^2(]0,T[;\mathbb{H}^{-1}(\mathcal{O}))$.
- The initial function $a(x)\in\mathbb{L}^2(\mathcal{O})$.
- $\sigma^{\varepsilon}(t,x)=\sigma(t,x,\tau=\frac{t}{\varepsilon^{\gamma}},y=\frac{x}{\varepsilon})$ is periodic with respect to both local variables τ and y.

The problem $\mathscr{P}b1$ admits a unique weak solution $u^{\varepsilon}\in\mathbb{L}^2(]0,T[;\mathbb{H}^1_0(\mathcal{O}))\cap\mathscr{C}([0,T],\mathbb{L}^2(\mathcal{O}))$.

The global or homogenized solution $u_0(t,x)$ associated to $\mathscr{P}b$ is

$$
\widetilde{\mathscr{P}b}=\begin{cases}u_0(t,x)\in\mathbb{L}^2(0,T;\mathbb{H}^1_0(\mathcal{O})),\\[2mm] \dfrac{\mathrm{d}u_0}{\mathrm{d}t}(t,x)-\dfrac{\partial}{\partial x}\left[\left\{\displaystyle\iint_{\mathbb{S}^2_{\#}}\sigma(t,x,\tau,y)\left[1+\dfrac{\partial\chi}{\partial y}(\tau,y)\right]\mathrm{d}\tau\,\mathrm{d}y\right\}\dfrac{\partial u_0}{\partial x}(t,x)\right]=f(t,x) \quad \forall (t,x)\in\mathbb{S}^2\\[2mm] u_0(t=0,x)=a(x) \quad \forall x\in\mathcal{O},\end{cases}
$$

when $\sigma^{\varepsilon}(t,x)$ is of the form $\sigma^{\varepsilon}(t,x)=\sigma(t/\varepsilon^{\gamma},x/\varepsilon)$, $\widetilde{\mathscr{P}b}$ reduces to

$$
\widetilde{\widetilde{\mathscr{P}b}}=\begin{cases}u_0(t,x)\in\mathbb{L}^2(]0,T[;\mathbb{H}^1_0(\mathcal{O})),\\[2mm] \dfrac{du_0}{\mathrm{d}t}(t,x)-\left\{\displaystyle\iint_{\mathbb{S}^2_{\#}}\sigma(\tau,y)\left[1+\dfrac{\partial\chi}{\partial y}(\tau,y)\right]\mathrm{d}\tau\,\mathrm{d}y\right\}\dfrac{\partial^2 u_0}{\partial x^2}(t,x)=f(t,x) \quad \forall (t,x)\in\mathbb{S}^2,\\[2mm] u_0(t=0,x)=a(x) \quad \forall x\in\mathcal{O}.\end{cases}
$$

The periodic function $\chi(\tau,y)$ which depends only on the local variables τ and y is related to the correction term $u_1(t,x,\tau,y)$ through the relation

$$
u_1(t,x,\tau,y)=\frac{\partial u_0}{\partial x}(t,x)\chi(\tau,y), \quad \text{where } \chi\in\mathbb{L}^2_{\#}(]0,T[;\mathbb{H}^1_{\#}/\mathbb{R}). \tag{3}
$$

To close the homogenized problem $\widetilde{\mathscr{P}b}$ or $\widetilde{\widetilde{\mathscr{P}b}}$, a relation satisfied by $\chi(\tau,y)$ is required. As proved in the appendix, three cases involving the real parameter γ are in order for the closure. It yields

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the following local equations:

- **Case 1: $0<\gamma<2$**

$$\begin{cases} \dfrac{\partial}{\partial y}\left\{\sigma(t,x,\tau,y)\left[1+\dfrac{\partial\chi}{\partial y}(\tau,y)\right]\right\}=0 \\ \text{or} \\ \dfrac{\partial}{\partial y}\left\{\sigma(\tau,y)\left[1+\dfrac{\partial\chi}{\partial y}(\tau,y)\right]\right\}=0. \end{cases} \tag{4}$$

- **Case 2: $\gamma=2$**

$$\begin{cases} \dfrac{d\chi}{d\tau}(\tau,y)-\dfrac{\partial}{\partial y}\left\{\sigma(t,x,\tau,y)\left[1+\dfrac{\partial\chi}{\partial y}(\tau,y)\right]\right\}=0 \\ \text{or} \\ \dfrac{d\chi}{d\tau}(\tau,y)-\dfrac{\partial}{\partial y}\left\{\sigma(\tau,y)\left[1+\dfrac{\partial\chi}{\partial y}(\tau,y)\right]\right\}=0 \end{cases} \tag{5}$$

- **Case 3: $\gamma>2$**

$$\begin{cases} \dfrac{\partial}{\partial y}\left\{\left(\displaystyle\int_0^1 \sigma(t,x,\tau,y)\,d\tau\right)\left[1+\dfrac{\partial\kappa}{\partial y}(y)\right]\right\}=0 \\ \text{or} \\ \dfrac{\partial}{\partial y}\left\{\left(\displaystyle\int_0^1 \sigma(\tau,y)\,d\tau\right)\left[1+\dfrac{\partial\kappa}{\partial y}(y)\right]\right\}=0 \end{cases} \tag{6}$$

3. SPECTRAL STOCHASTIC FORMULATION

Let $(\Omega,\mathscr{F},\mathscr{P})$ be a probability space. As in [18], we denote by $\mathscr{H}=\mathbb{L}^2(\Omega,\mathscr{F},\mathscr{P})$ the Hilbert space of square integrable functions on Ω with inner product

$$\mathbb{E}\{f\ g\}=\int_\Omega f(\omega)g(\omega)\mathscr{P}(d\omega) \quad \forall f,g\in\mathscr{H}=\mathbb{L}^2(\Omega,\mathscr{F},\mathscr{P})$$

and norm $\{\mathbb{E}\{f^2\}\}^{1/2}$. We define the following Hilbert space $\mathbb{L}^2(]0,T[;\mathbb{L}^2(\mathscr{O};\mathscr{H}))$ as the space of functions $t\longmapsto v(t)$ from $]0,T[\longmapsto\mathbb{L}^2(\mathscr{O};\mathscr{H})$, which are measurable and which satisfy

$$|||u|||_{\mathbb{L}^2(]0,T[;\mathbb{L}^2(\mathscr{O};\mathscr{H}))}=\left\{\int_0^T \|u(t)\|^2_{\mathbb{L}^2(\mathscr{O};\mathscr{H})}\,dt\right\}^{1/2},$$

here $\|u(t)\|_{\mathbb{L}^2(\mathscr{O};\mathscr{H})}=\left\{\int_\mathscr{O}|u(t,x)|^2_\mathscr{H}\,dx\right\}^{1/2}$ and $|u(t,x)|_\mathscr{H}=[\mathbb{E}\{u^2(t,x)\}]^{1/2}=\left\{\int_\Omega u^2(t,x,\omega)\mathscr{P}(d\omega)\right\}^{1/2}$. Similarly, we define $\mathbb{L}^2(]0,T[;\mathbb{H}^1(\mathscr{O};\mathscr{H}))$ the Hilbert space endowed with the inner product

$$\begin{aligned}((u,v))&=(u,v)+(\nabla u,\nabla v)\\ &=\iint_{\mathbb{S}^2}\left[\int_\Omega u(t,x,\omega)v(t,x,\omega)\mathscr{P}(d\omega)\right]dt\,dx+\iint_{\mathbb{S}^2}\left[\int_\Omega \nabla u(t,x,\omega)\nabla v(t,x,\omega)\mathscr{P}(d\omega)\right]dt\,dx.\end{aligned}$$

Let $\mathscr{V}$ be a closed subset of $\mathbb{H}^1(\mathscr{O};\mathscr{H})$ containing $\mathbb{H}^1_0(\mathscr{O};\mathscr{H})$. For $T>0$ fixed, $t\in]0,T[$a.e., let $b(t;u,v):\mathscr{V}\times\mathscr{V}\longmapsto\mathbb{R}$ the bilinear form defined by

$$b(t;u,v)=\iint_{\mathbb{S}^2}\left[\int_\Omega \sigma^\varepsilon(t,x,\omega)\frac{\partial u}{\partial x}(t,x,\omega)\frac{\partial v}{\partial x}(t,x,\omega)\mathscr{P}(d\omega)\right]dt\,dx.$$

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Under the assumptions

- $\forall t \in]0, T[$ and $\forall \omega \in \Omega, \sigma^{\varepsilon}(t, x, \omega) \in \mathbb{L}^{\infty}(]0; T[; \mathbb{L}^2(\mathcal{O}; \mathscr{H}))\forall x \in \mathcal{O}$
- $\forall t \in]0, T[\forall x \in \mathcal{O}, \exists \beta > 0,$ such that $\sigma^{\varepsilon}(t, x, \omega) > \beta$ almost surely

the bilinear form $b(t; u, v)$ satisfies the following properties:

- the function $t \longmapsto b(t; u, v)$ is measurable $\forall u, v \in \mathscr{V}$,
- $|b(t; u, v)| \leqslant c_1 \|u\| \|v\|$ $t \in]0, T[$ a.e. $\forall u, v \in \mathscr{V}$,
- $b(t; v, v) \geqslant c_2 \|v\|^2 - c_3 |v|^2$ $t \in]0, T[$ a.e. $\forall v \in \mathscr{V}$,

where c_1, c_2, and c_3 are constants. Furthermore, for a given $f \in \mathbb{L}^2(]0, T[; \mathscr{V}')$, $\mathscr{V}'$ being the dual space of $\mathscr{V}$, the problem

$$\begin{cases} \text{find } u^{\varepsilon} \in \mathbb{L}^2(]0, T[; \mathscr{V}) \\ \displaystyle\iint_{\mathbb{S}^2} \int_{\Omega} \frac{du^{\varepsilon}}{\mathrm{d}t}(t, x, \omega) v \mathscr{P}(\mathrm{d}\omega) \mathrm{d}t \, \mathrm{d}x + b(t; u^{\varepsilon}, v) = \iint_{\mathbb{S}^2} \int_{\Omega} f(t, x) v \mathscr{P}(\mathrm{d}\omega) \mathrm{d}t \, \mathrm{d}x, \; t \in]0, T[\text{a.e.} \\ \quad \forall v \in \mathscr{V}, \\ u^{\varepsilon}(t=0, x, \omega) = a(x), \end{cases}$$

admits a unique solution $u^{\varepsilon} \in \mathbb{L}^2(]0, T[; \mathscr{V})$. Note that the above problem is the weak formulation of

$$\begin{cases} \dfrac{\mathrm{d}u^{\varepsilon}}{\mathrm{d}t}(t, x, \omega) - \dfrac{\partial}{\partial x}\left\{\sigma^{\varepsilon}(t, x, \omega) \dfrac{\partial u^{\varepsilon}}{\partial x}(t, x, \omega)\right\} = f(t, x) \quad \forall (t, x, \omega) \in]0, T[\times \mathcal{O} \times \Omega, \\ u^{\varepsilon}(t, x, \omega) = 0 \quad \forall (t, x, \omega) \in]0, T[\times \partial\mathcal{O} \times \Omega, \\ u^{\varepsilon}(t=0, x, \omega) = a(x) \quad \forall (x, \omega) \in \mathcal{O} \times \Omega. \end{cases}$$

Following the Cameron & Martin theorem [19], any stochastic process $u(t, x, \omega) \in \mathbb{L}^2(]0, T[; \mathbb{L}^2(\mathcal{O}; \mathscr{H}))$ or $\mathbb{L}^2(]0, T[; \mathbb{H}^1(\mathcal{O}; \mathscr{H}))$ can be represented as

$$u(t, x, \omega) = \sum_{i=0}^{\infty} u_i(t, x) \Phi_i(\omega), \tag{7}$$

where the functionals $\{\Phi_i(\omega)\}_{i=0,1,\ldots}$ are the generalized Hermite polynomials also known as Weiner-chaos polynomials [20, 21]. We exploit the spectral stochastic representation (7) to introduce the chaos two-scale convergence.

Definition 1

A sequence of stochastic processes $u^{\varepsilon}(x, \omega)$ which is said to chaos two-scale converges to a stochastic process $u(x, y, \omega)$ if

$$\lim_{\varepsilon \searrow 0} \iint_{\mathcal{O} \times \Omega} u^{\varepsilon}(x, \omega) \psi\left(x, \frac{x}{\varepsilon}, \omega\right) \mathscr{P}(\mathrm{d}\omega) \, \mathrm{d}x = \iiint_{\mathcal{O} \times Y \times \Omega} u(x, y, \omega) \psi(x, y, \omega) \mathscr{P}(\mathrm{d}\omega) \mathrm{d}y \, \mathrm{d}x$$

$$\forall \psi \in \mathscr{D}(\mathcal{O}; \mathscr{C}^{\infty}_{\#}(Y; \mathscr{H}))$$

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By virtue of the stochastic representation (7), the above definition can be formally justified as follows:

$$\lim_{N\to\infty}\lim_{\varepsilon\searrow 0}\iint_{\mathscr{O}\times\Omega}\left\{\sum_{i=0}^{N}u_i^{\varepsilon}(x)\Phi_i(\omega)\right\}\psi\left(x,\frac{x}{\varepsilon},\omega\right)\mathscr{P}(\mathrm{d}\omega)\,\mathrm{d}x$$

$$=\lim_{N\to\infty}\sum_{i=0}^{N}\left\{\lim_{\varepsilon\searrow 0}\iint_{\mathscr{O}\times\Omega}u_i^{\varepsilon}(x)\psi(x,\frac{x}{\varepsilon},\omega)\mathscr{P}(\mathrm{d}\omega)\,\mathrm{d}x\right\}\Phi_i(\omega)$$

$$=\lim_{N\to\infty}\sum_{i=0}^{N}\left\{\iiint_{\mathscr{O}\times Y\times\Omega}u_i(x,y)\psi(x,y,\omega)\mathscr{P}(\mathrm{d}\omega)\,\mathrm{d}y\,\mathrm{d}x\right\}\Phi_i(\omega)$$

$$=\iint_{\mathscr{O}\times Y\times\Omega}\lim_{N\to\infty}\left\{\sum_{i=0}^{N}u_i(x,y)\Phi_i(\omega)\right\}\psi(x,y,\omega)\mathscr{P}(\mathrm{d}\omega)\,\mathrm{d}y\,\mathrm{d}x$$

$$=\iiint_{\mathscr{O}\times Y\times\Omega}u(x,y,\omega)\psi(x,y,\omega)\mathscr{P}(\mathrm{d}\omega)\,\mathrm{d}y\,\mathrm{d}x.$$

It should be pointed out that the $\lim_{N\to\infty}$ has to be taken with respect to the topology described by Holden *et al.* [22].

The theorems of Nguetseng [9] and Allaire [11] can then be extended to the stochastic case by using the polynomial chaos framework; it results in the following two claims:

Claim 1
Let u^{ε} be a uniformly bounded sequence in $\mathbb{L}^2(\mathscr{O};\mathscr{H})$. Then there exist a subsequence from ε, still denoted by ε, and a stochastic process $u_0(x,y,\omega)\in\mathbb{L}^2(\mathscr{O};\mathbb{L}^2_{\#}(Y;\mathscr{H}))$, such that $u^{\varepsilon}(x,\omega)$ chaos two-scale converges to $u_0(x,y,\omega)$ almost surely.

Claim 2
Let u^{ε} be a sequence of functions uniformly bounded in $\mathbb{H}^1(\mathscr{O};\mathscr{H})$. Then there exists a subsequence from ε, still denoted by ε, such that,

$$u^{\varepsilon}(x,\omega)\rightharpoonup u_0(x,\omega)\ \text{weakly in}\ \mathbb{H}^1(\mathscr{O};\mathscr{H}),\ \text{almost surely as}\ \varepsilon\searrow 0,$$

and there exists a stochastic process $u_1=u_1(x,y,\omega)\in\mathbb{L}^2(\mathscr{O};H^1_{\#}(Y;\mathscr{H}))$, such that,

$$\nabla u^{\varepsilon}(x,\omega)\ \text{chaos two-scale converges to}\ \nabla_x u_0(x,\omega)+\nabla_y u_1(x,y,\omega),\ \text{almost surely as}\ \varepsilon\searrow 0.$$

3.1. Random forcing

We consider the stochastic homogenization problem

$$\widetilde{\mathscr{P}b}2:\begin{cases}\dfrac{\mathrm{d}u^{\varepsilon}}{\mathrm{d}t}(t,x,\omega)-\dfrac{\partial}{\partial x}\left[\sigma^{\varepsilon}(t,x)\dfrac{\partial u^{\varepsilon}}{\partial x}(t,x,\omega)\right]=f(x,\omega)\quad\forall(t,x,\omega)\in]0,T[\times\mathscr{O}\times\Omega,\\[2ex] u^{\varepsilon}(t,x,\omega)=0\quad\forall(t,x,\omega)\in]0,T[\times\partial\mathscr{O}\times\Omega,\\[2ex] u^{\varepsilon}(t=0,x,\omega)=a(x)\quad\forall(x,\omega)\in\mathscr{O}\times\Omega,\end{cases}$$

where the randomness is introduced into the problem through the stochastic process $f(x,\omega)$. We assume that the covariance kernel $\mathscr{K}(x,y)$ of $f(x,\omega)$ is known, and that the random process $f(x,\omega)$ has a mean $\bar{f}(x)$ and a finite variance, $E[f(x,\omega)-\bar{f}(x)]^2$, that is bounded for all $x\in\mathscr{O}$. According to Van Trees [23], the process can then be expressed as

$$f(x,\omega)=\bar{f}(x)+\sum_{i=1}^{\infty}\sqrt{\lambda_i}\,g_i(x)\xi_i(\omega),\tag{8}$$

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in which λ_i and $g_i(x)$ are the eigenvalues and eigenfunctions of the covariance function $\mathscr{K}(x,y)$, i.e. the solution of the homogeneous Fredholm integral equation of the second kind

$$\int_{\mathcal{O}} \mathscr{K}(x,y) g_i(x)\,\mathrm{d}x = \lambda_i g_i(y). \tag{9}$$

For practical implementation, the sum in (8) can be approximated by a finite number of terms that is optimal; that is, the mean square approximation error is minimized as shown in [21, 24]. The Karhunen–Loève decomposition (8) becomes

$$f(x,\omega) = \bar{f}(x) + \sum_{i=1}^{\mathrm{KL}} \sqrt{\lambda_i}\, g_i(x)\xi_i(\omega). \tag{10}$$

The covariance function of the solution process $u^{\varepsilon}(t,x,\omega)$ being not known *a priori*, a Karhunen–Loève expansion cannot be used to represent it. A truncated polynomial chaos expansion can be used to represent the solution process.

$$u^{\varepsilon}(t,x,\omega) = \sum_{j=0}^{P} u_j^{\varepsilon}(t,x)\Phi_j(\omega). \tag{11}$$

The spectral stochastic formulation of problem $\widetilde{\mathscr{P}b}2$ is then

$$\begin{cases} \displaystyle\sum_{j=0}^{P} \frac{\mathrm{d}u_j^{\varepsilon}}{\mathrm{d}t}(t,x) - \frac{\partial}{\partial x}\left[\sigma^{\varepsilon}(t,x)\frac{\partial u_j^{\varepsilon}}{\partial x}(t,x)\right]\Phi_j(\omega) = \bar{f}(x) + \sum_{i=1}^{\mathrm{KL}} \sqrt{\lambda_i}\, g_i(x)\xi_i(\omega), \\ u^{\varepsilon}(t,x,\omega) = 0 \quad \forall (t,x,\omega) \in]0,T[\times \partial\mathcal{O} \times \Omega, \\ u^{\varepsilon}(t=0,x,\omega) = a(x) \quad \forall (x,\omega) \in \mathcal{O} \times \Omega, \end{cases}$$

which can be reduced to

$$\begin{cases} \text{for } k = 0,\dots,P, \\ \displaystyle\frac{\mathrm{d}u_k^{\varepsilon}}{\mathrm{d}t}(t,x) - \frac{\partial}{\partial x}\left[\sigma^{\varepsilon}(t,x)\frac{\partial u_k^{\varepsilon}}{\partial x}(t,x)\right] = \frac{\langle \bar{f}(x)\Phi_k(\omega)\rangle}{\langle \Phi_k(\omega)\Phi_k(\omega)\rangle} + \sum_{i=1}^{\mathrm{KL}} \sqrt{\lambda_i}\, g_i(x) \frac{\langle \xi_i(\omega)\Phi_k(\omega)\rangle}{\langle \Phi_k(\omega)\Phi_k(\omega)\rangle}, \\ u^{\varepsilon}(t,x,\omega) = 0 \quad \forall (t,x,\omega) \in]0,T[\times \partial\mathcal{O} \times \Omega, \\ u^{\varepsilon}(t=0,x,\omega) = a(x) \quad \forall (x,\omega) \in \mathcal{O} \times \Omega. \end{cases}$$

In the light of Section 2, and by employing the chaos two-scale convergence, the stochastic homogenized problem to $\widetilde{\mathscr{P}b}2$ is

$$\widetilde{\mathscr{P}b}2 = \begin{cases} \forall (t,x,\omega) \in]0,T[\times \mathcal{O} \times \Omega, \text{ and for } k = 0,\dots,P, \\ \displaystyle\frac{\mathrm{d}u_{0,k}}{\mathrm{d}t}(t,x) - \frac{\partial}{\partial x}\left[\tilde{\sigma}\frac{\partial u_{0,k}}{\partial x}(t,x)\right] = \frac{\langle \bar{f}(x)\Phi_k(\omega)\rangle}{\langle \Phi_k(\omega)\Phi_k(\omega)\rangle} + \sum_{i=1}^{\mathrm{KL}} \sqrt{\lambda_i}\, g_i(x) \frac{\langle \xi_i(\omega)\Phi_k(\omega)\rangle}{\langle \Phi_k(\omega)\Phi_k(\omega)\rangle}, \\ u_0(t=0,x) = a(x) \quad \forall x \in \mathcal{O}, \end{cases}$$

where $\tilde{\sigma}$ is given by

$$\begin{cases} \tilde{\sigma} = \tilde{\sigma}(t,x) = \displaystyle\iint_{\mathbb{S}_{\#}^2} \sigma(t,x,\tau,y)\left[1 + \frac{\partial \chi}{\partial y}(\tau,y)\right]\mathrm{d}\tau\,\mathrm{d}y \quad \text{if } 0 < \gamma \leqslant 2, \\ \tilde{\sigma} = \tilde{\sigma}(t,x) = \displaystyle\int_Y \left(\int_0^1 \sigma(t,x,\tau,y)\,\mathrm{d}\tau\right)\left[1 + \frac{\partial \kappa}{\partial y}(y)\right]\mathrm{d}y \quad \text{if } \gamma > 2. \end{cases}$$

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The periodic correction functions $\chi(\tau, y)$ and $\kappa(y)$ satisfy

$$\begin{cases} \dfrac{\partial}{\partial y}\left\{\sigma(t,x,\tau,y)\left[1+\dfrac{\partial\chi}{\partial y}(\tau,y)\right]\right\}=0 & \text{if } 0<\gamma<2,\\ \dfrac{\mathrm{d}\chi}{\mathrm{d}\tau}(\tau,y)-\dfrac{\partial}{\partial y}\left\{\sigma(t,x,\tau,y)\left[1+\dfrac{\partial\chi}{\partial y}(\tau,y)\right]\right\}=0 & \text{if } \gamma=2,\\ \dfrac{\partial}{\partial y}\left\{\left(\displaystyle\int_0^1 \sigma(t,x,\tau,y)\,\mathrm{d}\tau\right)\left[1+\dfrac{\partial\kappa}{\partial y}(y)\right]\right\}=0 & \text{if } \gamma>2. \end{cases}$$

3.2. *Random material property*

We turn our attention now to the case where the material property σ^ε is a stochastic process,

$$\sigma^\varepsilon=\sigma^\varepsilon(t,x,\omega) \quad \text{for } (t,x,\omega)\in]0,T[\times\mathcal{O}\times\Omega.$$

The stochastic homogenization problem to be solved is

$$\widetilde{\mathscr{P}b}3: \begin{cases} \dfrac{\mathrm{d}u^\varepsilon}{\mathrm{d}t}(t,x,\omega)-\dfrac{\partial}{\partial x}\left(\sigma^\varepsilon(t,x,\omega)\dfrac{\partial u^\varepsilon}{\partial x}(t,x,\omega)\right)=f(t,x) \quad \forall (t,x,\omega)\in]0,T[\times\mathcal{O}\times\Omega,\\ u^\varepsilon(t,x,\omega)=0 \quad \forall(t,x,\omega)\in]0,T[\times\partial\mathcal{O}\times\Omega,\\ u^\varepsilon(t=0,x,\omega)=a(x) \quad \forall(x,\omega)\in\mathcal{O}\times\Omega. \end{cases}$$

We make the assumption that $\sigma^\varepsilon(t,x,\omega)$ is of the form

$$\sigma^\varepsilon(t,x,\omega)=\alpha^\varepsilon(x)\beta^\varepsilon(t,\omega)=\alpha\left(\frac{x}{\varepsilon}\right)\beta\left(\frac{t}{\varepsilon^\gamma},\omega\right)=\alpha(y)\beta(\tau,\omega),$$

and that the random process $\beta(\tau,\omega)$ is defined on the probability space $(\Omega,\mathscr{F},P)$ and indexed on $[0,1]$. Let $\mathscr{R}(\tau_1,\tau_2)$ be the covariance kernel of $\beta(\tau,\omega)$, then the process can be expressed as

$$\beta(\tau,\omega)=\bar{\beta}(\tau)+\sum_{i=1}^{\infty}\sqrt{\mu_i}h_i(\tau)\xi_i(\omega), \tag{12}$$

where μ_i and h_i are solutions to the eigenvalue problem

$$\int_0^1 \mathscr{R}(\tau_1,\tau_2)h_i(\tau_1)\,\mathrm{d}\tau=\mu_i h_i(\tau_2).$$

Subsequently, the stochastic process $\sigma^\varepsilon(t,x,\omega)$ becomes

$$\sigma^\varepsilon(t,x,\omega)=\sigma(\tau,y,\omega)=\alpha(y)\left\{\bar{\beta}(\tau)+\sum_{i=1}^{\infty}\sqrt{\mu_i}h_i(\tau)\xi_i(\omega)\right\}. \tag{13}$$

The spectral stochastic formulation of $\widetilde{\mathscr{P}b}3$ reads as

$$\begin{cases} \displaystyle\sum_{j=0}^{P}\frac{\mathrm{d}u_j^\varepsilon}{\mathrm{d}t}(t,x)\Phi_j(\omega)-\frac{\partial}{\partial x}\left[\sum_{j=0}^{P}\sum_{i=0}^{\mathrm{KL}}\sqrt{\mu_i}\alpha(y)h_i(\tau)\frac{\partial u_j^\varepsilon}{\partial x}(t,x)\xi_i(\omega)\Phi_j(\omega)\right]=f(t,x),\\ u^\varepsilon(t,x,\omega)=0 \quad \forall(t,x,\omega)\in]0,T[\times\partial\mathcal{O}\times\Omega,\\ u^\varepsilon(t=0,x,\omega)=a(x) \quad \forall(x,\omega)\in\mathcal{O}\times\Omega, \end{cases}$$

where μ_0 and $h_0^\varepsilon(t)$ have been set to 1 and $\bar{\beta}^\varepsilon(t)$, respectively, and where only a finite number, KL, of Gaussian random variables have been employed.

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We multiply the first equation of the above formulation by $\Phi_k(\omega)$, and take the average; it results as

$$\begin{cases} \text{for } k=0,1,\ldots,P, \\ \dfrac{\mathrm{d}u_k^{\varepsilon}}{\mathrm{d}t}(t,x)-\sum_{j=0}^{P}\sum_{i=0}^{\mathrm{KL}}\sqrt{\mu_i}h_i(\tau)\dfrac{\partial}{\partial x}\left[\alpha(y)\dfrac{\partial u_j^{\varepsilon}}{\partial x}(t,x)\right]\dfrac{\langle \xi_i(\omega)\Phi_j(\omega)\Phi_k(\omega)\rangle}{\langle \Phi_k(\omega)\Phi_k(\omega)\rangle}=\dfrac{\langle f(t,x)\Phi_k(\omega)\rangle}{\langle \Phi_k(\omega)\Phi_k(\omega)\rangle}, \\ u^{\varepsilon}(t,x,\omega)=0 \quad \forall (t,x,\omega)\in]0,T[\times\partial\mathcal{O}\times\Omega, \\ u^{\varepsilon}(t=0,x,\omega)=a(x) \quad \forall (x,\omega)\in\mathcal{O}\times\Omega. \end{cases}$$

From the chaos two-scale convergence, we derive the stochastic homogenized problem of $\widetilde{\mathscr{P}b}3$.

$$\widetilde{\mathscr{P}b}3=\begin{cases} \text{for } k=0,\ldots,P, \\ \dfrac{\mathrm{d}u_{0,k}}{\mathrm{d}t}(t,x)-\dfrac{\partial}{\partial x}\left[\sum_{i=0}^{P}\sum_{j=0}^{P}\tilde{\sigma}_j(t,x)\left\{\dfrac{\partial u_0^i}{\partial x}(t,x)\right\}\dfrac{\langle \Phi_i(\omega)\Phi_j(\omega)\Phi_k(\omega)\rangle}{\langle \Phi_k(\omega)\Phi_k(\omega)\rangle}\right]=\dfrac{\langle f(t,x)\Phi_k(\omega)\rangle}{\langle \Phi_k(\omega)\Phi_k(\omega)\rangle}, \\ \{u_0^k(t=0,x)\}=a(x) \quad \forall x\in\mathcal{O} \text{ if } k=0, \\ \{u_0^k(t=0,x)\}=0 \quad \forall x\in\mathcal{O} \text{ if } k\neq 0. \end{cases}$$

The chaos component $\tilde{\sigma}_k(t,x)$ of the stochastic process $\tilde{\sigma}(t,x,\omega)$ satisfies

$$\begin{cases} \tilde{\sigma}_k(t,x)=\sum_{i=0}^{\mathrm{KL}}\sqrt{\mu_i}\iint_{\mathbb{S}_{\#}^2}\alpha(y)h_i(\tau)\left[\langle \xi_i\Phi_k\rangle+\sum_{j=0}^{P}\dfrac{\partial \chi_j}{\partial y}(\tau,y)\langle \xi_i\Phi_j\Phi_k\rangle\right]\mathrm{d}\tau\,\mathrm{d}y \quad \text{if } 0<\gamma\leqslant 2, \\ \tilde{\sigma}_k(t,x)=\sum_{i=0}^{\mathrm{KL}}\sqrt{\mu_i}\int_Y\alpha(y)\left(\int_0^1 h_i(\tau)d\tau\right)\left[\langle \xi_i\Phi_k\rangle+\sum_{j=0}^{P}\dfrac{\partial \kappa_j}{\partial y}(y)\langle \xi_i\Phi_j\Phi_k\rangle\right]\mathrm{d}y \quad \text{if } \gamma>2. \end{cases}$$

The stochastic processes $\chi(\tau,y,\omega)$ and $\kappa(y,\omega)$ are solutions to

$$\begin{cases} \sum_{i=0}^{\mathrm{KL}}\sum_{j=0}^{P}h_i(\tau)\dfrac{\partial}{\partial y}\left\{\alpha(y)\dfrac{\partial \chi_j}{\partial y}(\tau,y)\right\}\langle \xi_i\Phi_j\Phi_k\rangle=-\sum_{i=0}^{\mathrm{KL}}\dfrac{\partial \alpha}{\partial y}(y)h_i(\tau)\langle \xi_i\Phi_k\rangle \quad \text{if } 0<\gamma<2, \\ \dfrac{\mathrm{d}\chi_k}{\mathrm{d}\tau}(\tau,y)-\sum_{i=0}^{\mathrm{KL}}\sum_{j=0}^{P}\sqrt{\mu_i}h_i(\tau)\dfrac{\partial}{\partial y}\left\{\alpha(y)\dfrac{\partial \chi_j}{\partial y}(\tau,y)\right\}\dfrac{\langle \xi_i\Phi_j\Phi_k\rangle}{\langle \Phi_k\Phi_k\rangle}=-\sum_{i=0}^{\mathrm{KL}}\sqrt{\mu_i}\dfrac{\partial \alpha}{\partial y}(y)h_i(\tau)\dfrac{\langle \xi_i\Phi_k\rangle}{\langle \Phi_k\Phi_k\rangle} \\ \quad \text{if } \gamma=2, \\ \sum_{i=0}^{\mathrm{KL}}\sum_{j=0}^{P}\dfrac{\partial}{\partial y}\left[\alpha(y)\dfrac{\partial \kappa_j}{\partial y}(y)\right]\langle \xi_i\Phi_j\Phi_k\rangle=-\sum_{i=0}^{\mathrm{KL}}\dfrac{\partial \alpha}{\partial y}(y)\langle \xi_i\Phi_k\rangle \quad \text{if } \gamma>2. \end{cases}$$

4. NUMERICAL PROCEDURE AND RESULTS

We present the numerical algorithm for the random material property case. The computational approach follows Jardak and Ghanem [25] and Jardak *et al.* [26]; it is based on the spectral collocation method for the spatial discretization [27, 28]. As observed by Engquist and Luo [29] and also pointed out in E [13], numerical dissipations tend to damp out the small scales, and the numerical dispersions tend to move the small scales to wrong locations and incorrectly account for their effects on the large scales. As numerical dissipation and dispersions are inherent to finite difference and finite element methods, the decision to employ the spectral method is then justified.

We start by solving for the chaos components of the processes $\chi(\tau, y, \omega)$ and $\kappa(y, \omega)$. As each of them is periodic with respect to the spatial direction y, a Fourier collocation method Canuto *et al.* [27] and Peyret [28] is employed.

For ω fixed, for each τ, and for $0<\gamma<2$ the vector form of the equation

$$\sum_{i=0}^{\mathrm{KL}} \sum_{j=0}^{P} h_i(\tau) \frac{\partial}{\partial y} \left\{ \alpha(y) \frac{\partial \chi_j}{\partial y}(\tau, y) \right\} \langle \xi_i \Phi_j \Phi_k \rangle = - \sum_{i=0}^{\mathrm{KL}} \frac{\partial \alpha}{\partial y}(y) h_i(\tau) \langle \xi_i \Phi_k \rangle$$

is given by

$$\begin{bmatrix} [A_{00}] & \cdots & [A_{0j}] & \cdots & [A_{0P}] \\ \vdots & & \vdots & & \vdots \\ [A_{k0}] & \cdots & [A_{kj}] & \cdots & [A_{kP}] \\ \vdots & & \vdots & & \vdots \\ [A_{P0}] & \cdots & [A_{Pj}] & \cdots & [A_{PP}] \end{bmatrix} \begin{bmatrix} [\vec{\chi}_0] \\ \vdots \\ [\vec{\chi}_k] \\ \vdots \\ [\vec{\chi}_P] \end{bmatrix} = \begin{bmatrix} [\vec{c}_0] \\ \vdots \\ [\vec{c}_k] \\ \vdots \\ [\vec{c}_P] \end{bmatrix}. \tag{14}$$

Here the vector

$$[\vec{\chi}_k] = \begin{bmatrix} \chi_k(\tau_1, y_1) \\ \chi_k(\tau_1, y_2) \\ \vdots \\ \chi_k(\tau_1, y_n) \\ \chi_k(\tau_2, y_1) \\ \chi_k(\tau_2, y_2) \\ \vdots \\ \chi_k(\tau_2, y_n) \\ \vdots \\ \chi_k(\tau_m, y_1) \\ \chi_k(\tau_m, y_2) \\ \vdots \\ \chi_k(\tau_m, y_n) \end{bmatrix} \tag{15}$$

is an approximation of the k-th chaos component as represented by its values at the collocation points,

$$\tau_m = \frac{2\pi(m-1)}{\varepsilon^{\gamma} M}, \quad m = 1, \ldots, M \quad \text{and} \quad y_n = \frac{2\pi(n-1)}{\varepsilon N}, \quad n = 1, \ldots, N.$$

Let D_M^{τ} and D_N^{y} denote the Fourier collocation differentiation matrices in the τ and y directions, respectively. We denote by

$$\mathbb{D}^{y} = I_{M \times M} \otimes D_N^{y} \quad \text{and} \quad \mathbb{D}^{\tau} = D_M^{\tau} \otimes I_{N \times N} \tag{16}$$

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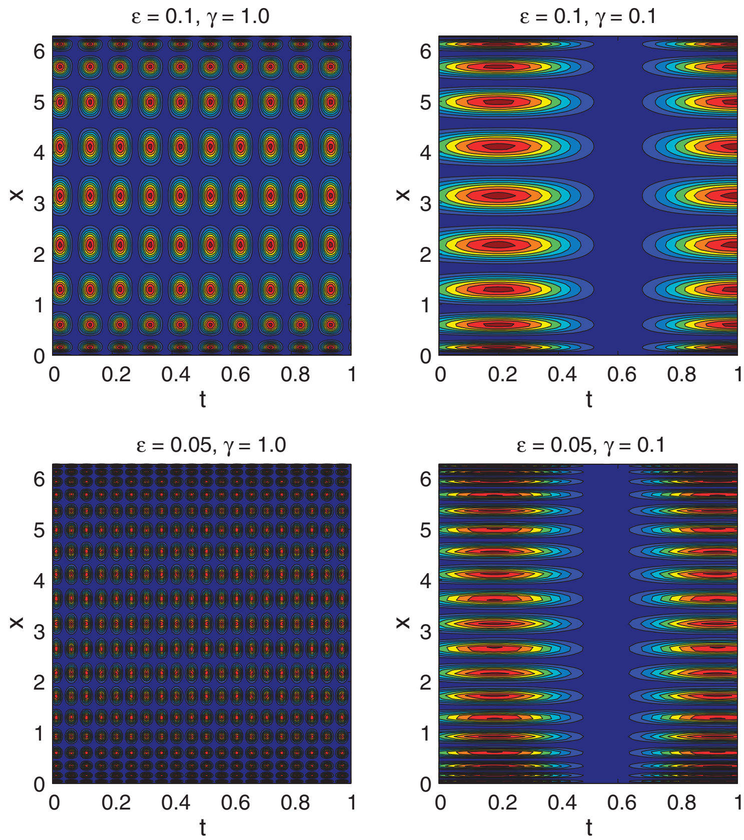

Figure 1. Material property for different ε and γ.

the tensor or kronecker products, the block matrix $[A_{kj}]$ can then be expressed as

$$[A_{kj}]_{j,k=0,\dots,P} = \sum_{i=0}^{\mathrm{KL}} h_i(\tau)\{\Gamma' \ \mathbb{D}^y + \Gamma \ (\mathbb{D}^y)^2\}\langle \xi_i \Phi_j \Phi_k\rangle, \tag{17}$$

Γ and Γ' are the diagonal matrices defined by $\Gamma_{i,i} = \alpha(y_i)$ and $\Gamma'_{i,i} = (\partial\alpha/\partial y)(y_i)$, respectively.

For $\gamma = 2$, the integration of

$$\frac{\mathrm{d}\chi_k}{\mathrm{d}\tau}(\tau, y) = \sum_{i=0}^{\mathrm{KL}} \sqrt{\mu_i} h_i(\tau)\left[\frac{\partial\alpha}{\partial y}(y)\frac{\langle \xi_i \Phi_k\rangle}{\langle \Phi_k \Phi_k\rangle} + \sum_{j=0}^{P} \frac{\partial}{\partial y}\left\{\alpha(y)\frac{\partial \chi_j}{\partial y}(\tau, y)\right\}\frac{\langle \xi_i \Phi_j \Phi_k\rangle}{\langle \Phi_k \Phi_k\rangle}\right] \tag{18}$$

requires a time discretization. As ξ is periodic in the τ direction, the numerical treatment of (18) is similar to the case where $0<\gamma<2$ with a change in $[A_{kj}]$. Now, the block matrix $[A_{kj}]$ is a follows:

$$[A_{kj}]_{j,k=0,\dots,P} = \mathbb{D}^\tau - \sum_{i=0}^{\mathrm{KL}} \sum_{j=0}^{P} \sqrt{\mu_i} h_i(\tau)\{\Gamma' \ \mathbb{D}^y + \Gamma \ (\mathbb{D}^y)^2\}\frac{\langle \xi_i \Phi_j \Phi_k\rangle}{\langle \Phi_k \Phi_k\rangle}. \tag{19}$$

The k-th right-hand side block $[\vec{c}_k]$ is given by

$$[\vec{c}_k]_{k=0,\dots,P} = \sum_{i=0}^{\mathrm{KL}} \sqrt{\mu_i} h_i(\tau)\left[\frac{\partial\alpha}{\partial y}(y)\frac{\langle \xi_i \Phi_k\rangle}{\langle \Phi_k \Phi_k\rangle}\right].$$

The numerical calculation of the homogenized chaos component $\tilde{\sigma}_k(t,x)$ of the process $\tilde{\sigma}(t,x,\omega)$ involves a numerical integration. To that end, a composite numerical quadrature of order 2 is employed in approximating the integrals.

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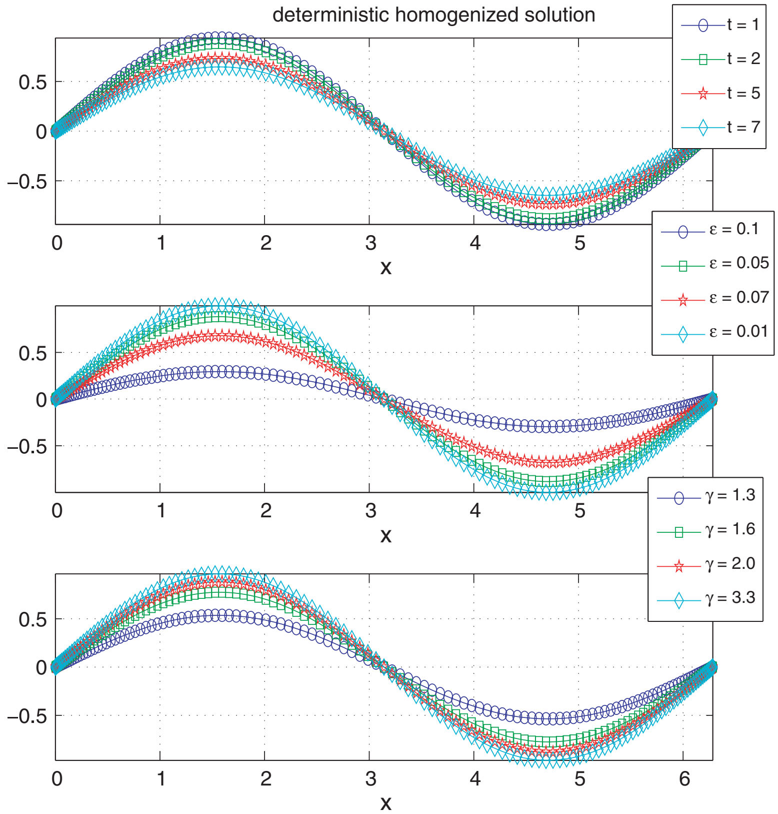

Figure 2. Deterministic homogenization for different ε, γ, and time t.

The numerical treatment of $\widetilde{\mathscr{P}b}3$ utilizes the Chebyshev collocation method, see Canuto *et al.* [27] to discretize the spatial global variable x along with the implicit Adams Moulton two-step method to advance in global time t.

Therefore, the approximation of the equation

$$\frac{\mathrm{d}u_0^k}{\mathrm{d}t}(t,x)-\frac{\partial}{\partial x}\left[\sum_{i=0}^{P}\sum_{j=0}^{P}\tilde{\sigma}_j(t,x)\frac{\partial u_0^i}{\partial x}(t,x)\frac{\langle\Phi_i(\omega)\Phi_j(\omega)\Phi_k(\omega)\rangle}{\langle\Phi_k(\omega)\Phi_k(\omega)\rangle}\right]=\frac{\langle f(t,x)\Phi_k(\omega)\rangle}{\langle\Phi_k(\omega)\Phi_k(\omega)\rangle}$$

is then

$$\{u_0^k\}^{n+1}-\{u_0^k\}^n=\frac{\Delta t}{12}[5\mathscr{L}(t^{n+1},x,u_0^k(t^{n+1},x))+8\mathscr{L}(t^n,x,u_0^k(t^n,x))-\mathscr{L}(t^{n-1},x,u_0^k(t^{n-1},x))] \tag{20}$$

with

$$\mathscr{L}(t,x,u_0^k(t,x))=\sum_{i=0}^{P}\sum_{j=0}^{P}\frac{\partial}{\partial x}\left[\tilde{\sigma}_j(t,x)\frac{\partial u_0^i}{\partial x}(t,x)\right]\frac{\langle\Phi_i(\omega)\Phi_j(\omega)\Phi_k(\omega)\rangle}{\langle\Phi_k(\omega)\Phi_k(\omega)\rangle}.$$

The Chebyshev–Gauss–Labatto (CGL) interpolation points

$$x(m)=\cos\left(\frac{(m-1)\pi}{N}\right)\quad\text{for } 1\leqslant m\leqslant N+1$$

have been used. along with the Chebyshev collocation derivative matrix $\mathscr{D}_N$ described in Trefethen [30].

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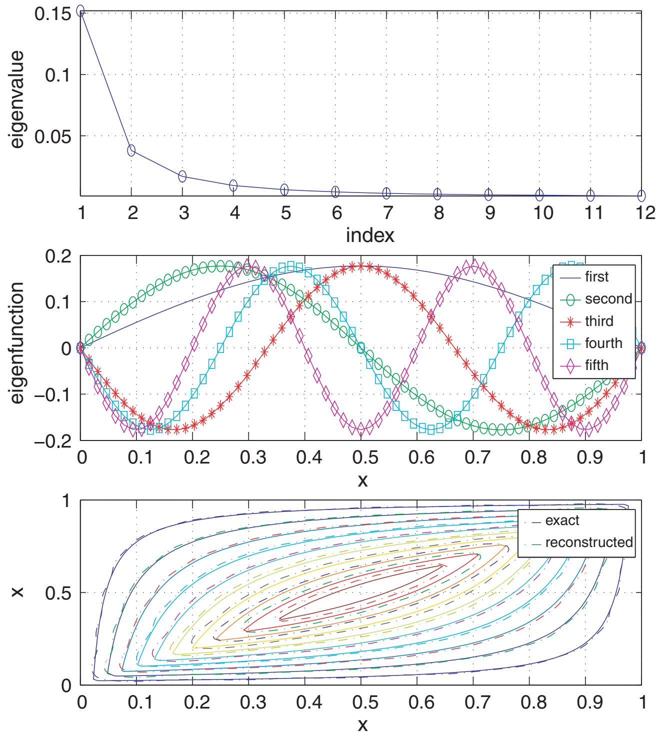

Figure 3. Eigenspectra and contours of the exact and the reconstructed of the Brown–Bridge covariance kernel.

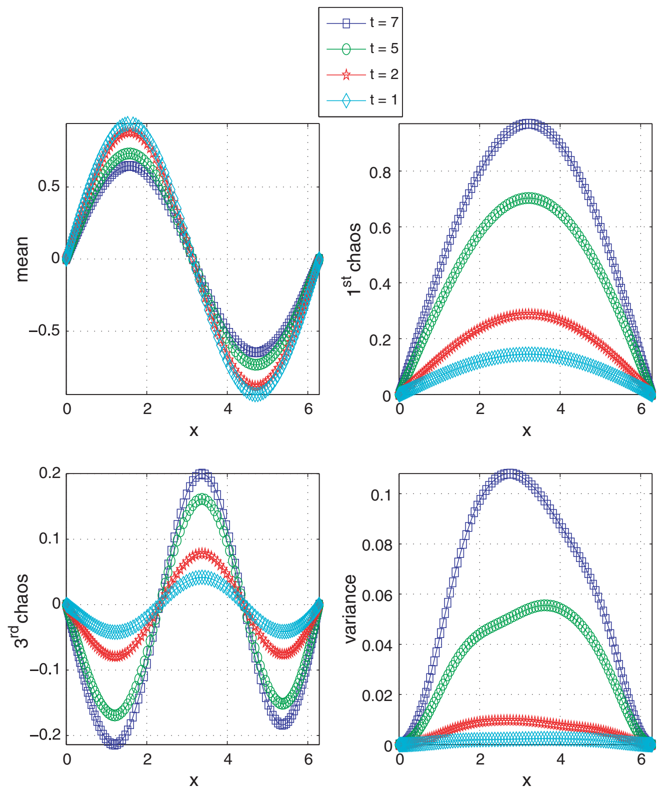

Figure 4. Stochastic forcing case: chaos components and variance for different time t, $\varepsilon=0.05$ and for $\gamma=2.0$.

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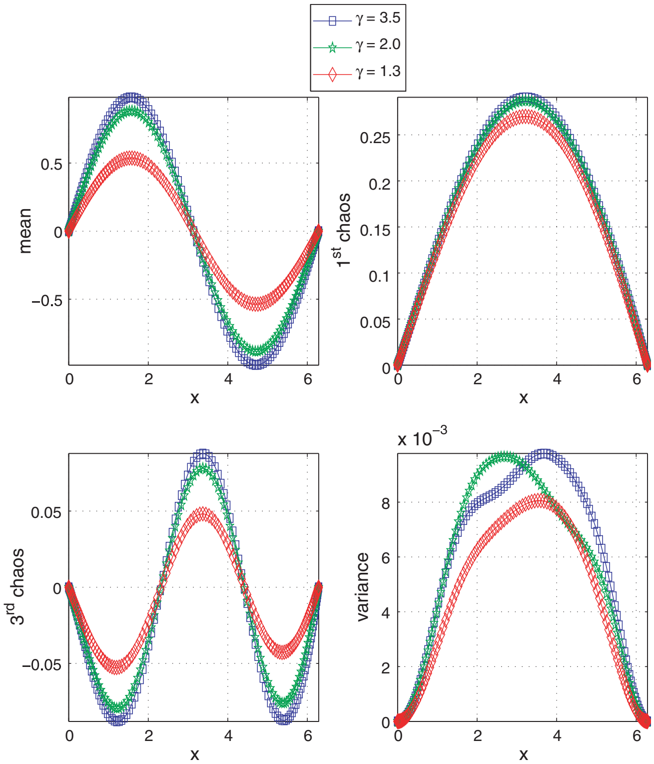

Figure 5. Stochastic forcing case: chaos components and variance for different γ, after $t=2$ and for $\varepsilon=0.05$.

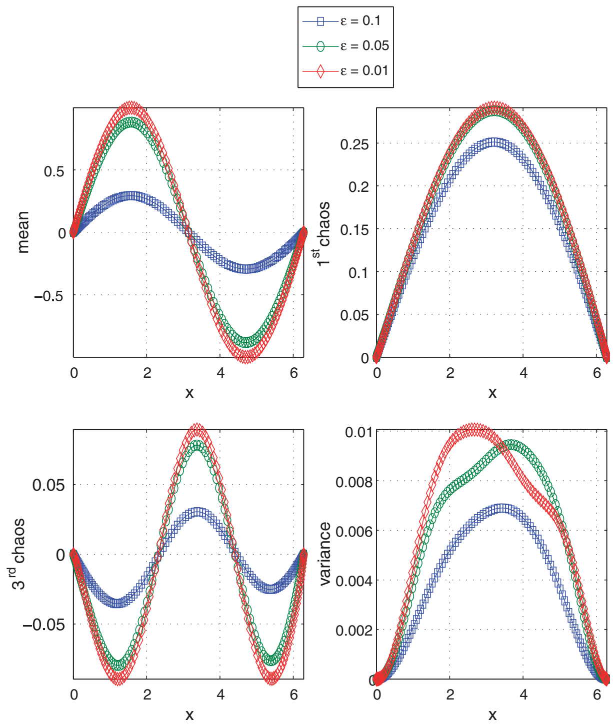

Figure 6. Stochastic forcing case: chaos components and variance for different ε, after $t=2$ and for $\gamma=2.0$.

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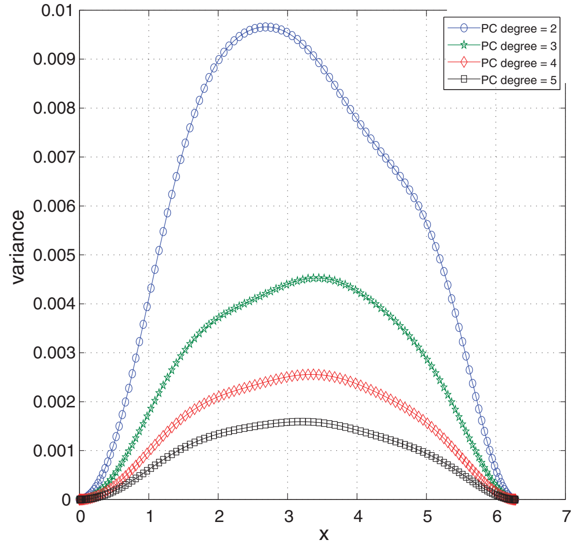

Figure 7. Stochastic forcing case: variances for different polynomial chaos degrees after $t=2$, and $\varepsilon=0.005$, $\gamma=2.0$.

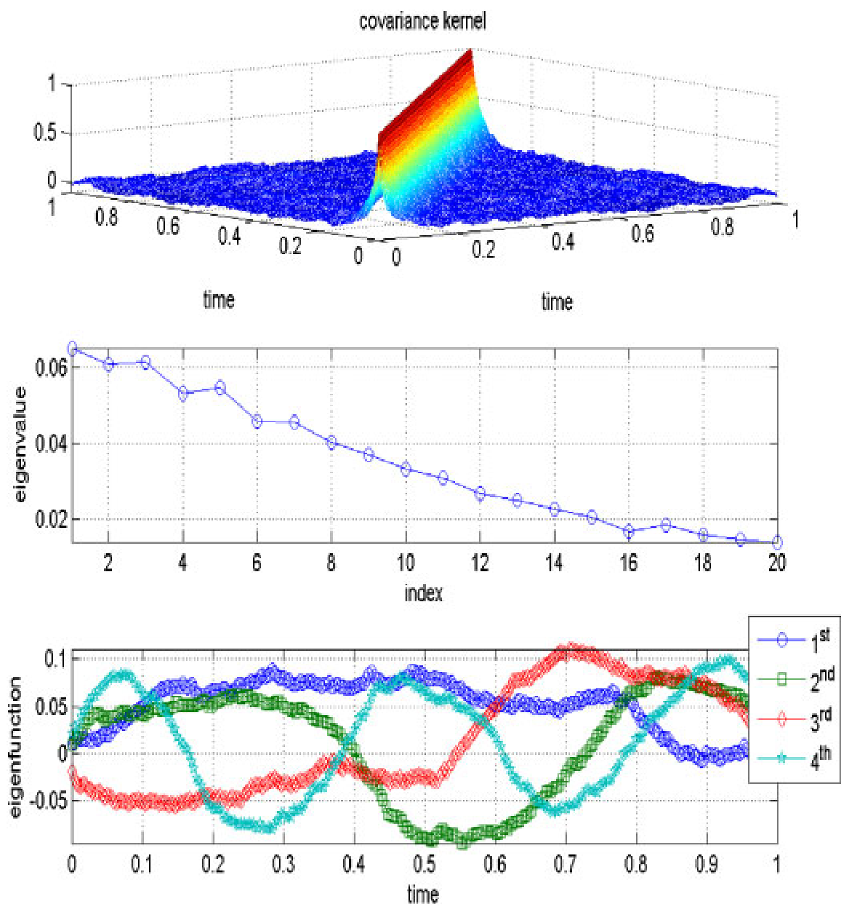

Figure 8. Spectral decomposition of an auto-regressive process of order 1.

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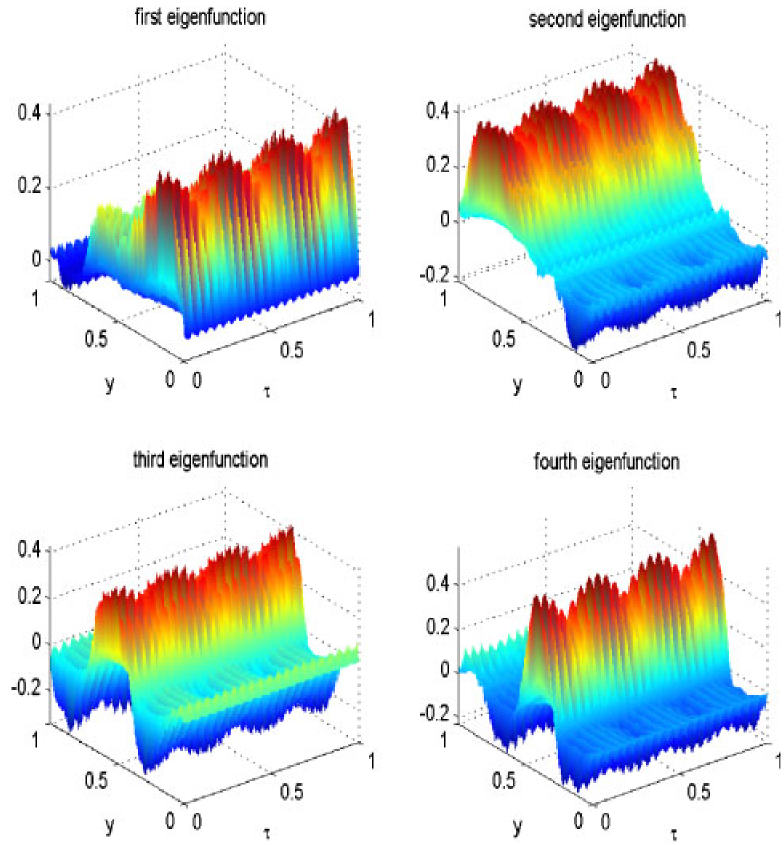

Figure 9. Space–time eigenfunctions after $t=1$.

We employ the following material property:

$$\sigma^{\varepsilon}(t,x)=\sigma\left(\frac{t}{\varepsilon^{\gamma}},\frac{x}{\varepsilon}\right)=\left(2.1+2\cos\left(2\pi\frac{x}{\varepsilon}\right)\right)\left(2.1+2\sin\left(2\pi\frac{t}{\varepsilon^{\gamma}}\right)\right)\quad\forall\ x\in[0,1]\quad\text{and}\quad 0<t\leqslant 1$$

and the initial solution $u^{\varepsilon}(t=0,x)=a(x)=\sin(2\pi x)\ \forall x\in[0,1]$. Neither ε nor γ is assumed random. For three different values of ε and for γ fixed, Figure 1 represents the deterministic material property $\sigma^{\varepsilon}(t,x)$ for different values of ε and γ. The contours present an insight into the complex structure of the material as ε decreases and γ increases.

Figure 2 presents the essentials of periodic deterministic homogenization. From the same figure one observes that for fixed values of $\varepsilon=0.05$, $\gamma=2.0$ and as time t increases from $t=1$ to $t=7$ the homogenized solution is diffusing. For a fixed time $t=2$ and fixed value of $\gamma=2.0$ the homogenized solution is again diffusing as the value of ε increases. The calculated homogenized solutions converge toward a numerical solution that will be taken as the effective or homogenized solution as ε decreases. In the meanwhile, for $t=2$ and $\varepsilon=0.05$ the homogenized solution is less diffusing as the time oscillating speed γ increases from $\gamma=1.3$ to $\gamma=3.3$. as the oscillating time speed increases. For the case of a random forcing, the non-stationary Brown-Bridge process is considered to represent $f(t,x,\omega)$. The Brown-Bridge process given by its covariance function

$$\mathscr{B}(x,y)=\min(x,y)-\frac{xy}{a}\quad\forall x,y\in[0,a]\tag{21}$$

has been used. In Figure 3 we provide the eigenvalues and the corresponding eigenfunctions of the Karhunen–Loeve decomposition. The contour plots of the theoretical (solid lines) and the reconstructed (dashed lines) covariance kernels are also shown in the same figure. It is worth noting

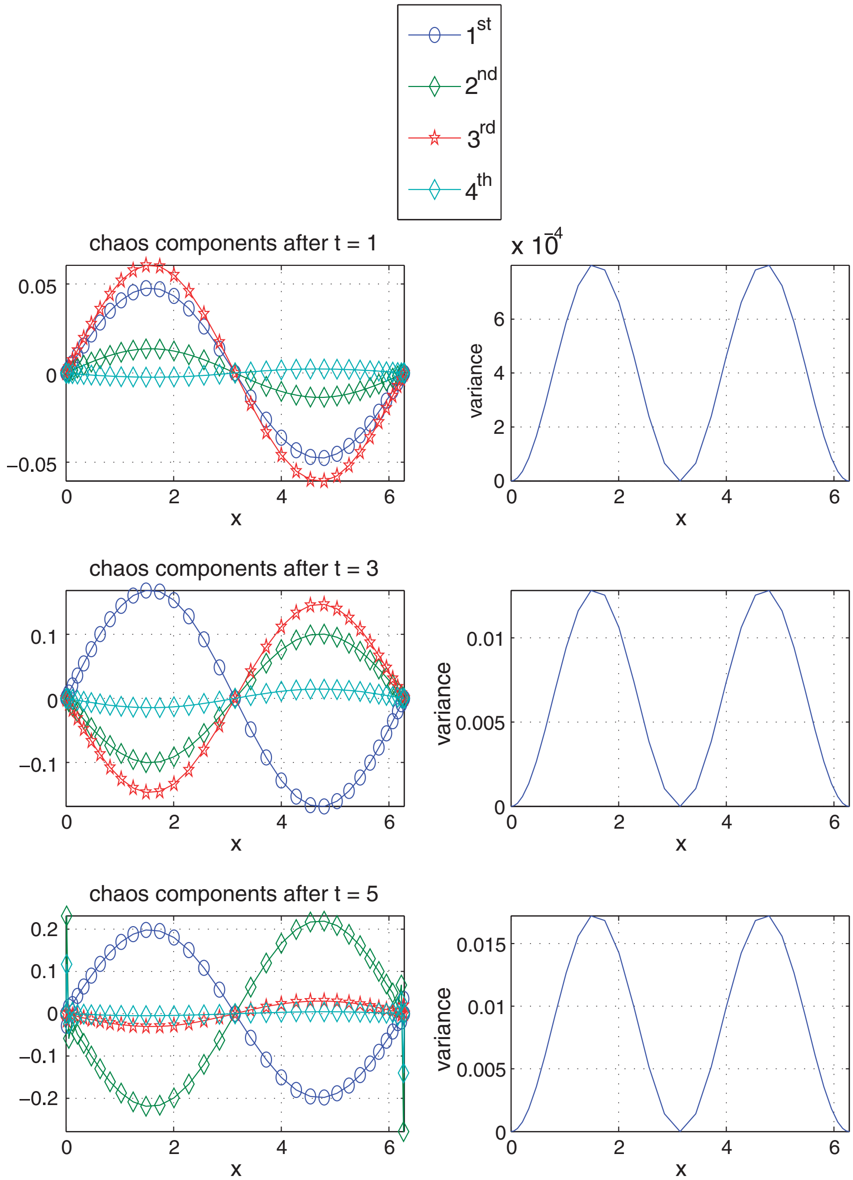

Figure 10. Stochastic random material property case: chaos components and variance, after $t=1$, $t=3$, and $t=5$, and $\varepsilon=0.005$, $\gamma=2.0$.

the good agreement between both kernels after a 12 terms truncation has been made. Now that

$$f(t,x,\omega)=f(x,\omega)=\underbrace{\bar{f}(x,\omega)}_{=0}+\sum_{i=1}^{3}\sqrt{\lambda_i}\,g_i(x)\xi_i \tag{22}$$

the solution to the homogenization problem is a stochastic process. In Figure 4, we present some of the chaos components including the mean. Here the values of ε and γ have been fixed whereas the time t varied. The mean solutions exhibit the same behavior as in the deterministic case. Indeed, as t increases, the mean solutions are more diffused. This behavior is not followed by the other chaos components as shown in the same figure. The variances are added to support the last statement.

In Figure 5, we fix the time t to 2 and ε to 0.05 and let γ vary. Again, similar to the deterministic case the means of the stochastic homogenized solutions are less diffusing as γ increases. The chaos components of the stochastic homogenized solutions and the variances all follow the same trend.

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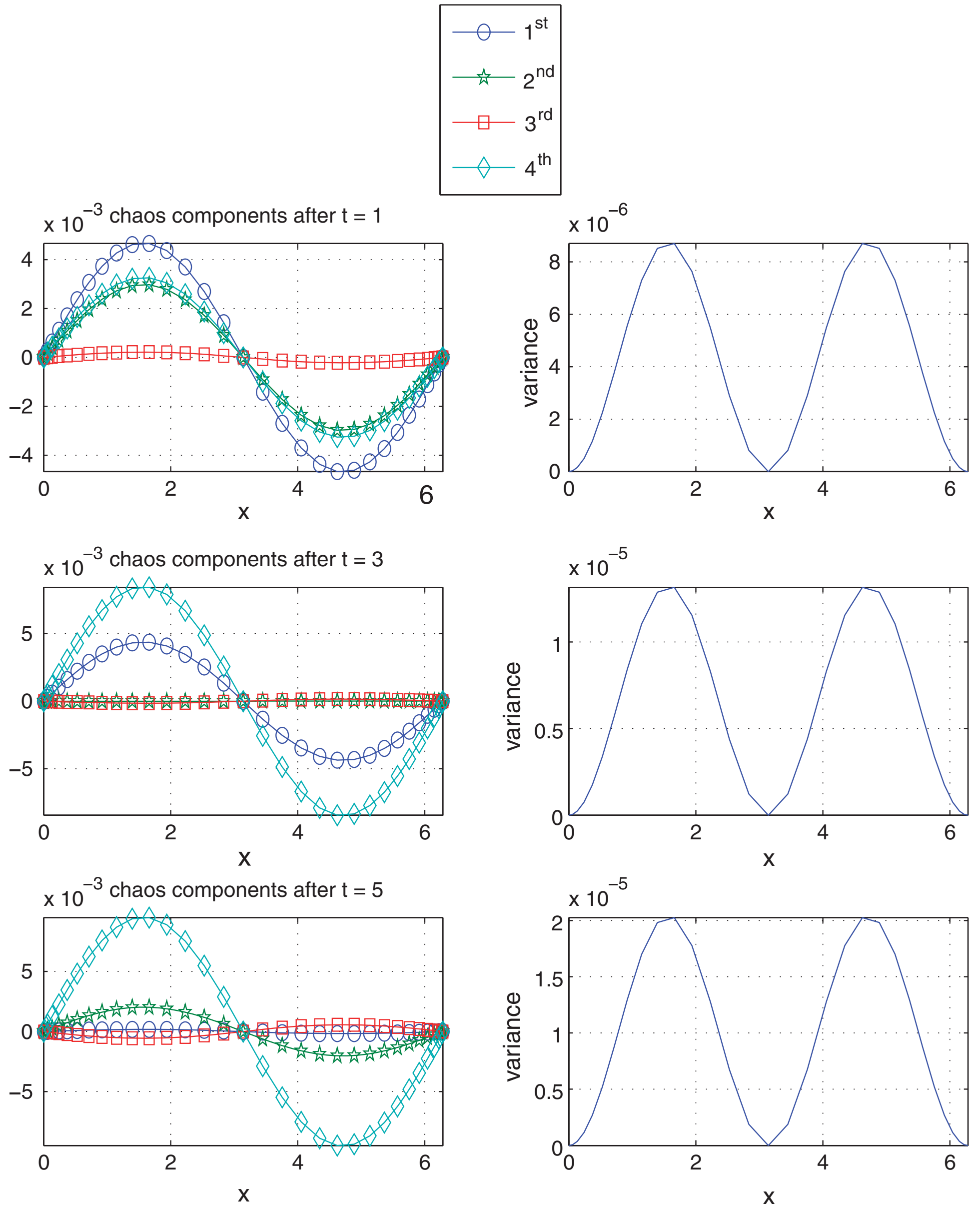

Figure 11. Stochastic random material property case: chaos components and variance, after $t=1$, $t=3$, and $t=5$, and $\varepsilon=0.005$, $\gamma=3.5$.

As ε increases whereas time t and the time oscillating speed γ are kept constant, the means of the stochastic homogenized solutions are more diffused and this is followed by the chaos components and the variances. This is depicted in Figure 6.

Finally, we present in Figure 7 the behavior of the variances as the degrees of the chaos polynomials are varied whereas the number of stochastic dimensions is kept equal to 3. The expected conclusion namely that the variance decreases with higher polynomial chaos degrees is well supported.

Following Section 3, we consider the stochastic material property of the form

$$\sigma^{\varepsilon}(t,x)=\alpha\left(\frac{x}{\varepsilon}\right)\beta\left(\frac{t}{\varepsilon^{\gamma}},\omega\right)$$

with $\alpha(x/\varepsilon)=2.1+2\cos(2\pi x/\varepsilon)$. No randomness in ε is assumed.

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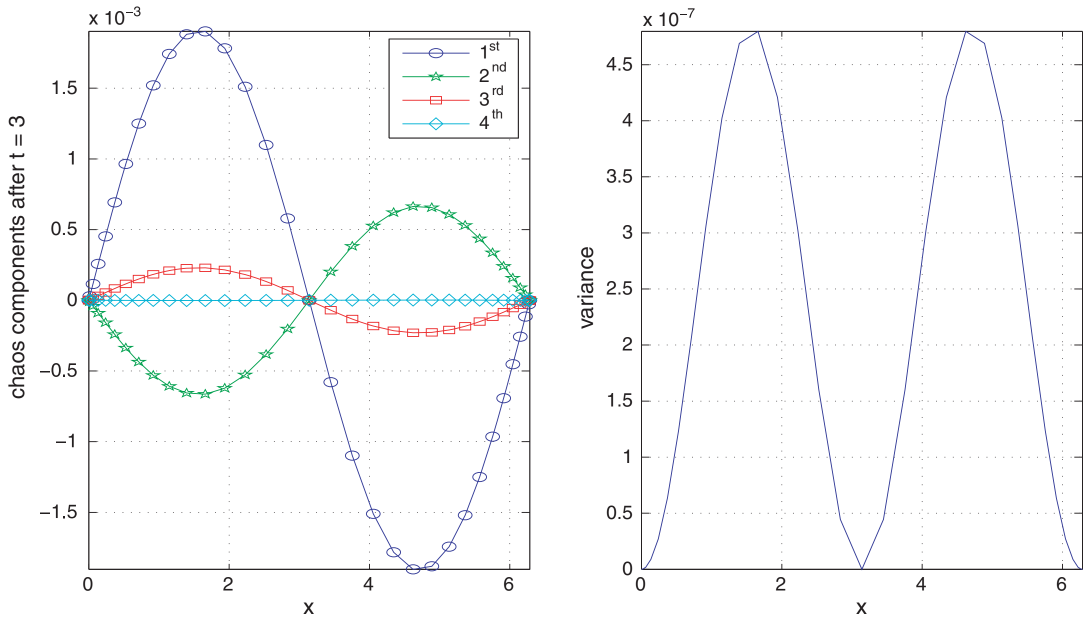

Figure 12. Stochastic random material property case: chaos components and variance after $t=3$, and $\varepsilon=0.001$, $\gamma=2$.

To represent the stochastic input $\beta((t/\varepsilon^{\gamma}),\omega)$, we consider a different way for correlations. Specifically, we use a discrete stationary input. We use the process

$$u_k = cu_{k-1} + af\xi_k. \tag{23}$$

The process (23) is auto-regressive of order 1 and corresponds to the Markov process [31]. The constant c is assumed to satisfy $|c|\leqslant 1$ to ensure that the process is of finite variance. Here, ξ_k is a random variable of mean zero and variance one, and f is a constant to be determined such that for the given values of a and c the variance of the process is equal to a^2. Using Monte Carlo simulations we construct numerically the variance kernel and subsequently extract the eigenvalues and eigenfunctions required for the input. The covariance kernel and the eigenspectrum are depicted in Figure 8. We set the mean value of $\beta(\tau)$ to $\bar{\beta}(\tau)=2.1+2\sin(2\pi(t/\varepsilon^{\gamma}))$, and multiply each eigenfunction of the Markov process by $2.1+2\cos(2\pi\frac{x}{\varepsilon})$. It results in the space–time eigenfunctions shown in Figure 9.

Figures 10–12 regroup chaos components of the stochastic homogenized process and variances for different setups. In Figure 10 the ε and γ are fixed and the time t varies. The setup for Figure 11 is similar to the one presented in (10) with the exception of $\gamma=3.5$ instead of $\gamma=2.0$. In Figure 12 the time is now fixed to $t=3$, $\varepsilon=0.01$, and $\gamma=2$. In all the above configurations, the amplitudes of the chaos components of the stochastic homogenized process are smaller but the trends are in agreement with those of the stochastic forcing case.

5. CONCLUSION

The present work dealt with random homogenization. A time evolution problem with a coefficient oscillating in both time and space with dissimilar speeds has been studied. Owing to the assumption of periodicity of $\sigma(t,(t/\varepsilon^{\gamma}),x,(x/\varepsilon))$ with respect to the fast variables, t/ε^{γ} and x/ε the two-scale convergence method of Nguetseng [9] and Allaire [10] has been employed to derive the homogenized problem instead of the laborious reiteration homogenization method of Bensoussan *et al.* [3]. Owing to the spectral stochastic decomposition's capability of separating the deterministic and random parts of a stochastic process, we extended the two-scale convergence to the stochastic

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case. A derivation of the spectral stochastic homogenization random forcing and random material have been achieved. Suitable numerical procedures have been devised and numerical results have been obtained and validated.

In the present work we have restricted our investigations to periodic homogenization, several theoretical extensions like the H and G convergence of Murat and Tartar [32] are available and have never been implemented in the framework of the present work. This opens new avenues for future research. We also envision extending the present work to the case where the scale ε is a random variable.

APPENDIX A

Nguetseng [9] presented a new concept of how to homogenize scales of partial differential equations, the so-called two-scale convergence method.

Definition 2
Let u^ε be a sequence of functions in $\mathbb{L}^2(\mathcal{O})$. u^ε two-scale converges to $u_0 = u_0(x, y)$ with $u_0 \in \mathbb{L}^2(\mathcal{O}, Y)$ if for any test function $\phi = \phi(x, y) \in \mathscr{D}(\mathcal{O}, \mathscr{C}^\infty_\#(Y))$

$$\lim_{\varepsilon \searrow 0} \int_\Omega u^\varepsilon(x)\phi\left(x, \frac{x}{\varepsilon}\right) \mathrm{d}x = \frac{1}{|Y|}\int_\Omega \int_Y u_0(x, y)\phi(x, y)\,\mathrm{d}y\,\mathrm{d}x.$$

The two-scale convergence method is an alternative to the so-called energy method of Tartar [33] for proving convergence in the case of periodic homogenization. It deals with convergence of integrals of the form

$$\int_\mathcal{O} u^\varepsilon(x)\phi\left(x, \frac{x}{\varepsilon}\right) \mathrm{d}x,$$

where $u^\varepsilon \in \mathbb{L}^2(\mathcal{O})$ and $\phi(x, y)$ is a smooth function periodic with respect to y. For the problem studied here, the following and more convenient versions of Nguetseng's theorems will be used in the sequel of the paper.

Theorem 1
Let $u^\varepsilon(t, x)$ be a uniformly bounded sequence in $\mathbb{L}^2(]0, T[; \mathbb{L}^2_{\mathrm{loc}}(\mathcal{O}))$. Then, there exist a subsequence of ε and a function $u_0(t, x, \tau, y) \in \mathbb{L}^2(]0, T[\times\mathcal{O}; \mathbb{L}^2_\#[0, 1] \times \mathbb{L}^2_\#(Y))$ such that u^ε two-scale converges to u_0.

The above theorem is a compactness result. It states that the two-scale limit u_0 is essentially the first term in the multiple scale expansion. The dependence of u_0 on the oscillations is granted through the auxiliary variables τ and y. In order to obtain more detailed information about the two-scale limit, uniform boundedness over the gradient of u^ε is required.

Theorem 2
Let $u^\varepsilon(t, x)$ be a uniformly bounded sequence in $\mathbb{L}^2(]0, T[; \mathbb{H}^1(\mathcal{O}))$. Then $u^\varepsilon(t, x)$ two-scale converges to a function $u_0(t, x) \in \mathbb{L}^2(]0, T[; \mathbb{H}^1(\mathcal{O}))$ and there exists a function $u_1(t, x, \tau, y) \in \mathbb{L}^2(]0, T[\times\mathcal{O}; \mathbb{L}^2_\#[0, 1] \times \mathbb{H}^1_\#(\mathcal{O})/\mathbb{R})$ such that, up to a sub sequence, $\nabla_x u^\varepsilon(t, x)$ two-scale converges to $\nabla_x u_0(t, x) + \nabla_y u_1(t, x, \tau, y)$. Moreover, $u_0(t, x)$ is the strong $\mathbb{L}^2(]0, T[\times\mathcal{O})$ limit of $u^\varepsilon(t, x)$.

We consider the second order parabolic partial differential equation

$$\mathscr{P}b = \begin{cases} \text{Find } u^\varepsilon(t, x) \\ \dfrac{\mathrm{d}u^\varepsilon}{\mathrm{d}t}(t, x) - \dfrac{\partial}{\partial x}\left\{\sigma^\varepsilon(t, x)\dfrac{\partial u^\varepsilon}{\partial x}(t, x)\right\} = f(t, x) \quad \forall (t, x) \in\ ]0, T[\times\mathcal{O} \\ u^\varepsilon(t, x) = 0 \quad \forall (t, x) \in\ ]0, T[\times\partial\mathcal{O} \\ u^\varepsilon(t = 0, x) = a(x) \quad \forall x \in \mathcal{O}. \end{cases}$$

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Under the following assumptions:

1. For $T>0$, $\sigma^{\varepsilon}(t,x)\in\mathbb{L}^{\infty}(]0,T[\times\mathscr{O})$, and, $\exists\beta>0$, such that, $\sigma^{\varepsilon}(t,x)\geqslant\beta$, for a.e. $t\in]0,T[$, and $x\in\mathscr{O}$.
2. $f\in\mathbb{L}^2(]0,T[;\mathbb{H}^{-1}(\mathscr{O}))$.
3. The initial function $a(x)\in\mathbb{L}^2(\mathscr{O})$.
5. $\sigma^{\varepsilon}(t,x)=\sigma(t,x,\tau=t/\varepsilon^{\gamma},y=x/\varepsilon)$ is periodic with respect to both local variables τ and y.

The problem $\mathscr{P}b$ admits a unique weak solution $u^{\varepsilon}\in\mathbb{L}^2(]0,T[;\mathbb{H}_0^1(\mathscr{O}))\cap\mathscr{C}([0,T],\mathbb{L}^2(\mathscr{O}))$. This is achieved by applying the theorem of Lions [34], Brezis [35], and Evans [36]. Furthermore, we have the following estimates:

$$\|u^{\varepsilon}\|_{\mathbb{L}^2(]0,T[;\mathbb{H}_0^1(\mathscr{O}))}+\|u^{\varepsilon}\|_{\mathscr{C}([0,T],\mathbb{L}^2(\mathscr{O}))}\leqslant C[\|a\|_{\mathbb{L}^2(\mathscr{O})}+\|f\|_{\mathbb{L}^2(]0,T[;\mathbb{H}^{-1}(\mathscr{O}))}], \tag{A1}$$

where the constant $C>0$ depends only on the diameter of $\mathscr{O}$. From the energy inequality (A1), we in particular deduce the uniform boundedness of $u^{\varepsilon}(t,x)$ in $\mathbb{L}^2(]0,T[;\mathbb{H}^1(\mathscr{O}))$ which enables us the use of Theorem 2.

In order to perform a homogenization procedure for $\mathscr{P}b$, we multiply the equation of $\mathscr{P}b1$ by a test function $\phi(t,x)\in\mathbb{L}^2_{\text{loc}}(]0,T[;\mathscr{D}(\mathscr{O}))$,

$$\iint_{\mathbb{S}^2}\left\{\frac{\mathrm{d}u^{\varepsilon}}{\mathrm{d}t}(t,x)-\frac{\partial}{\partial x}\left[\sigma\left(t,x,\frac{t}{\varepsilon^{\gamma}},\frac{x}{\varepsilon}\right)\frac{\partial u^{\varepsilon}}{\partial x}(t,x)\right]\right\}\phi(t,x)\,\mathrm{d}t\,\mathrm{d}x=\iint_{\mathbb{S}^2}f(t,x)\phi(t,x)\,\mathrm{d}t\,\mathrm{d}x,$$

since ϕ has a compact support, an integration by parts in both time and space yields

$$\begin{aligned}-\iint_{\mathbb{S}^2}u^{\varepsilon}(t,x)\frac{\mathrm{d}\phi}{\mathrm{d}t}(t,x)\,\mathrm{d}t\,\mathrm{d}x+\iint_{\mathbb{S}^2}\sigma\left(t,x,\frac{t}{\varepsilon^{\gamma}},\frac{x}{\varepsilon}\right)\frac{\partial u^{\varepsilon}}{\partial x}(t,x)\frac{\partial\phi}{\partial x}(t,x)\,\mathrm{d}t\,\mathrm{d}x\\ =\iint_{\mathbb{S}^2}f(t,x)\phi(t,x)\,\mathrm{d}t\,\mathrm{d}x+\int_{\mathscr{O}}a(x)\phi(0,x)\,\mathrm{d}x.\end{aligned} \tag{A2}$$

As $\varepsilon\searrow 0$, the application of Theorem 2 yields

$$\begin{aligned}-\iint_{\mathbb{S}^2}u_0(t,x)\frac{\mathrm{d}\phi}{\mathrm{d}t}(t,x)\,\mathrm{d}t\,\mathrm{d}x+\iiiint_{\mathbb{S}^4}\sigma(t,x,\tau,y)\left\{\frac{\partial u_0}{\partial x}(t,x)+\frac{\partial u_1}{\partial y}(t,x,\tau,y)\right\}\frac{\partial\phi}{\partial x}(t,x)\,\mathrm{d}\mu\\ =\iint_{\mathbb{S}^2}f(t,x)\phi(t,x)\,\mathrm{d}t\,\mathrm{d}x+\int_{\mathscr{O}}a(x)\phi(0,x)\,\mathrm{d}x.\end{aligned} \tag{A3}$$

Following Bensoussan *et al.* [3] and Cioranescu and Donato [6], the correction term $u_1(t,x,\tau,y)$ is then factorized using the separation of variables

$$u_1(t,x,\tau,y)=\frac{\partial u_0}{\partial x}(t,x)\chi(\tau,y)\quad\text{where } \chi\in\mathbb{L}^2_{\#}(]0,T[;\mathbb{H}^1_{\#}/\mathbb{R}). \tag{A4}$$

Equation (A3) yields

$$\begin{aligned}\iint_{\mathbb{S}^2}\left[-u_0(t,x)\frac{\mathrm{d}\phi}{\mathrm{d}t}(t,x)+\left\{\iint_{\mathbb{S}^2_{\#}}\sigma(t,x,\tau,y)\left[1+\frac{\partial\chi}{\partial y}(\tau,y)\right]\mathrm{d}\tau\,\mathrm{d}y\right\}\frac{\partial u_0}{\partial x}(t,x)\frac{\partial\phi}{\partial x}(t,x)\right]\mathrm{d}t\,\mathrm{d}x\\ =\iint_{\mathbb{S}^2}f(t,x)\phi(t,x)\,\mathrm{d}t\,\mathrm{d}x+\int_{\mathscr{O}}a(x)\phi(0,x)\,\mathrm{d}x.\end{aligned} \tag{A5}$$

As a result of integration by parts of (A5), we obtain

$$\begin{aligned}\iint_{\mathbb{S}^2}\left\{\frac{du_0}{\mathrm{d}t}(t,x)-\frac{\partial}{\partial x}\left[\left\{\iint_{\mathbb{S}^2_{\#}}\sigma(t,x,\tau,y)\left[1+\frac{\partial\chi}{\partial y}(\tau,y)\right]\mathrm{d}\tau\,\mathrm{d}y\right\}\frac{\partial u_0}{\partial x}(t,x)\right]\right\}\phi(t,x)\,\mathrm{d}t\,\mathrm{d}x\\ =\iint_{\mathbb{S}^2}f(t,x)\phi(t,x)\,\mathrm{d}t\,\mathrm{d}x\quad\forall\phi\in\mathbb{L}^2_{\text{loc}}(]0,T[;\mathscr{D}(\mathscr{O})).\end{aligned} \tag{A6}$$

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Because ϕ is arbitrary, the global or homogenized solution $u_0(t,x)$ of (A6) is

$$\widetilde{\mathscr{P}b}=\begin{cases} u_0(t,x)\in\mathbb{L}^2(0,T;\mathbb{H}_0^1(\mathscr{O})),\\ \dfrac{du_0}{\mathrm{d}t}(t,x)-\dfrac{\partial}{\partial x}\left[\left\{\displaystyle\iint_{\mathbb{S}^2_\#}\sigma(t,x,\tau,y)\left[1+\dfrac{\partial\chi}{\partial y}(\tau,y)\right]\mathrm{d}\tau\,\mathrm{d}y\right\}\dfrac{\partial u_0}{\partial x}(t,x)\right]=f(t,x)\forall(t,x)\in\mathbb{S}^2,\\ u_0(t=0,x)=a(x)\quad\forall x\in\mathscr{O}.\end{cases}$$

In the case where $\sigma^\varepsilon(t,x)$ is of the form $\sigma^\varepsilon(t,x)=\sigma(\frac{t}{\varepsilon^\gamma},\frac{x}{\varepsilon})$, $\widetilde{\mathscr{P}b}$ reduces to

$$\widetilde{\widetilde{\mathscr{P}b}}=\begin{cases} u_0(t,x)\in\mathbb{L}^2(]0,T[;\mathbb{H}_0^1(\mathscr{O})),\\ \dfrac{du_0}{\mathrm{d}t}(t,x)-\left\{\displaystyle\iint_{\mathbb{S}^2_\#}\sigma(\tau,y)\left[1+\dfrac{\partial\chi}{\partial y}(\tau,y)\right]\mathrm{d}\tau\,\mathrm{d}y\right\}\dfrac{\partial^2 u_0}{\partial x^2}(t,x)=f(t,x)\quad\forall(t,x)\in\mathbb{S}^2,\\ u_0(t=0,x)=a(x)\quad\forall x\in\mathscr{O}.\end{cases}$$

It is worth noting that in both cases, the local and global variables appear together in the homogenized equations.

To close the homogenized problem $\widetilde{\mathscr{P}b}$, a relation satisfied by $u_1(t,x,\tau,y)$ and subsequently by $\chi(\tau,y)$ is required. To this end, we consider the following form of the test function $\phi(t,x)$

$$\phi(t,x)=\phi\left(t,x,\frac{t}{\varepsilon^\gamma},\frac{x}{\varepsilon}\right)=\varepsilon^{\gamma-1}\phi_1\left(t,\frac{t}{\varepsilon^\gamma}\right)\phi_2\left(x,\frac{x}{\varepsilon}\right),\tag{A7}$$

where $\phi_1\in\mathbb{L}^2(]0,T[;\mathbb{L}^2_\#[0,1])$, $\phi_2\in\mathbb{H}_0^1(\mathscr{O},\mathbb{H}^1_\#(Y))$. This choice is suggested by Tartar's method of oscillating test functions see Murat and Tartar [32]. Equation (A2) then becomes

$$\begin{aligned}\iint_{\mathbb{S}^2}f(t,x)\phi(t,x)\,\mathrm{d}t\,\mathrm{d}x&=-\varepsilon^{\gamma-1}\iint_{\mathbb{S}^2}u^\varepsilon(t,x)\left\{\frac{\partial\phi_1}{\partial t}\left(t,\frac{t}{\varepsilon^\gamma}\right)+\frac{1}{\varepsilon^\gamma}\frac{\partial\phi_1}{\partial\tau}\left(t,\frac{t}{\varepsilon^\gamma}\right)\right\}\phi_2\left(x,\frac{x}{\varepsilon}\right)\mathrm{d}t\,\mathrm{d}x\\ &+\varepsilon^{\gamma-1}\iint_{\mathbb{S}^2}\sigma\left(t,x,\frac{t}{\varepsilon^\gamma},\frac{x}{\varepsilon}\right)\frac{\partial u^\varepsilon}{\partial x}(t,x)\left\{\frac{\partial\phi_2}{\partial x}\left(x,\frac{x}{\varepsilon}\right)+\frac{1}{\varepsilon}\frac{\partial\phi_2}{\partial y}\left(x,\frac{x}{\varepsilon}\right)\right\}\phi_1\left(t,\frac{t}{\varepsilon^\gamma}\right)\mathrm{d}t\,\mathrm{d}x.\end{aligned}\tag{A8}$$

Similarly, (A3) becomes

$$\begin{aligned}\iint_{\mathbb{S}^2}f(t,x)\phi(t,x)\,\mathrm{d}t\,\mathrm{d}x.&=-\varepsilon^{\gamma-1}\iint_{\mathbb{S}^2}u_0(t,x)\left\{\frac{\partial\phi_1}{\partial t}(t,\tau)+\frac{1}{\varepsilon^\gamma}\frac{\partial\phi_1}{\partial\tau}(t,\tau)\right\}\phi_2(x,y)\,\mathrm{d}t\,\mathrm{d}x\\ &+\varepsilon^{\gamma-1}\iiiint_{\mathbb{S}^4}\sigma(t,x,\tau,y)\left[\frac{\partial u_0}{\partial x}(t,x)+\frac{\partial u_1}{\partial y}(t,x,\tau,y)\right]\left\{\frac{\partial\phi_2}{\partial x}(x,y)+\frac{1}{\varepsilon}\frac{\partial\phi_2}{\partial y}(x,y)\right\}\phi_1(t,\tau)\mathrm{d}\mu.\end{aligned}\tag{A9}$$

Subtracting Equation (A9) from Equation (A8) yields

$$\begin{aligned}&\varepsilon^{\gamma-2}\left[\iint_{\mathbb{S}^2}\sigma\left(t,x,\frac{t}{\varepsilon^\gamma},\frac{x}{\varepsilon}\right)\frac{\partial u^\varepsilon}{\partial x}\frac{\partial\phi_2}{\partial y}\phi_1\,\mathrm{d}t\,\mathrm{d}x-\iiiint_{\mathbb{S}^4}\sigma(t,x,\tau,y)\left\{\frac{\partial u_0}{\partial x}+\frac{\partial u_1}{\partial y}\right\}\frac{\partial\phi_2}{\partial y}\phi_1\,\mathrm{d}\mu\right]\\ &+\varepsilon^{\gamma-1}\left[\iint_{\mathbb{S}^2}\sigma\left(t,x,\frac{t}{\varepsilon^\gamma},\frac{x}{\varepsilon}\right)\frac{\partial u^\varepsilon}{\partial x}\frac{\partial\phi_2}{\partial x}\phi_1\,\mathrm{d}t\,\mathrm{d}x-\iiiint_{\mathbb{S}^4}\sigma(t,x,\tau,y)\left\{\frac{\partial u_0}{\partial x}+\frac{\partial u_1}{\partial y}\right\}\frac{\partial\phi_2}{\partial x}\phi_1\,\mathrm{d}\mu\right]\\ &+\varepsilon^{\gamma}\left[\iint_{\mathbb{S}^2}\left\{\frac{u^\varepsilon(t,x)-u_0(t,x)}{\varepsilon}\right\}\left\{\frac{\partial\phi_1}{\partial t}(t,\tau)+\frac{1}{\varepsilon^\gamma}\frac{\partial\phi_1}{\partial\tau}(t,\tau)\right\}\phi_2(x,y)\,\mathrm{d}t\,\mathrm{d}x\right]=0.\end{aligned}\tag{A10}$$

From (A10), local equations will be derived by examining the three cases $\gamma-2<0$, $\gamma-2=0$, and, $\gamma-2>0$, respectively.

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Case 1: $0<\gamma<2$

The passage to the two-scale limit in (A10) gives

$$\iiiint_{\mathbb{S}^4} \sigma(t,x,\tau,y)\left[\frac{\partial u_0}{\partial x}(t,x)+\frac{\partial u_1}{\partial y}(t,x,\tau,y)\right]\frac{\partial \phi_2}{\partial y}(x,y)\phi_1(t,\tau)\,\mathrm{d}\mu$$
$$-\iiiint_{\mathbb{S}^4}\left\{\iint_{\mathbb{S}^2_\#}\sigma(t,x,\tau,y)\left[\frac{\partial u_0}{\partial x}(t,x)+\frac{\partial u_1}{\partial y}(t,x,\tau,y)\right]\frac{\partial \phi_2}{\partial y}(x,y)\phi_1(t,\tau)\,\mathrm{d}\tau\,\mathrm{d}y\right\}d\mu=0. \tag{A11}$$

As $\left\{\iint_{\mathbb{S}^2_\#}\sigma(t,x,\tau,y)\left[\frac{\partial u_0}{\partial x}(t,x)+\frac{\partial u_1}{\partial y}(t,x,\tau,y)\right]\frac{\partial \phi_2}{\partial y}(x,y)\phi_1(t,\tau)\,\mathrm{d}\tau\,\mathrm{d}y\right\}$ depends only on the global variables t, x, and, because of the periodicity over $Y\times[0,1]$,

$$\iiiint_{\mathbb{S}^4}\left\{\iint_{\mathbb{S}^2_\#}\sigma(t,x,\tau,y)\left[\frac{\partial u_0}{\partial x}(t,x)+\frac{\partial u_1}{\partial y}(t,x,\tau,y)\right]\frac{\partial \phi_2}{\partial y}(x,y)\phi_1(t,\tau)\,\mathrm{d}\tau\,\mathrm{d}y\right\}\mathrm{d}\mu=0.$$

Therefore (A11) reduces to

$$\iiiint_{\mathbb{S}^4} \sigma(t,x,\tau,y)\left[\frac{\partial u_0}{\partial x}(t,x)+\frac{\partial u_1}{\partial y}(t,x,\tau,y)\right]\frac{\partial \phi_2}{\partial y}(x,y)\phi_1(t,\tau)\,\mathrm{d}\mu=0.$$

The use of the separation of variables (A4), and the integration by parts with respect to the local variable y imply

$$\iint_{\mathbb{S}^2}\left[\iint_{\mathbb{S}^2_\#}\frac{\partial}{\partial y}\left[\sigma(t,x,\tau,y)\left\{1+\frac{\partial \chi}{\partial y}(\tau,y)\right\}\right]\phi_2(x,y)\phi_1(t,\tau)\,\mathrm{d}\tau\,\mathrm{d}y\right]\frac{\partial u_0}{\partial x}(t,x)\,\mathrm{d}t\,\mathrm{d}x=0 \quad \forall \phi_1,\phi_2.$$

Hence, for $0<\gamma<2$, the local equation is:

$$\frac{\partial}{\partial y}\left\{\sigma(t,x,\tau,y)\left[1+\frac{\partial \chi}{\partial y}(\tau,y)\right]\right\}=0. \tag{A12}$$

Case 2: $\gamma=2$

We replace γ by 2 in (A10). The dependency of

$$\left\{\iint_{\mathbb{S}^2_\#}\sigma(t,x,\tau,y)\left[\frac{\partial u_0}{\partial x}(t,x)+\frac{\partial u_1}{\partial y}(t,x,\tau,y)\right]\frac{\partial \phi_2}{\partial y}(x,y)\phi_1(t,\tau)\,\mathrm{d}\tau\,\mathrm{d}y\right\}$$

on the global variables t, x only, implies

$$\iint_{\mathbb{S}^2}\left\{\frac{u^\varepsilon(t,x)-u_0(t,x)}{\varepsilon}\right\}\frac{\partial \phi_1}{\partial \tau}(t,\tau)\phi_2(x,y)\,\mathrm{d}t\,\mathrm{d}x$$
$$+\iiiint_{\mathbb{S}^4}\sigma(t,x,\tau,y)\left[\frac{\partial u_0}{\partial x}(t,x)+\frac{\partial u_1}{\partial y}(t,x,\tau,y)\right]\frac{\partial \phi_2}{\partial y}(x,y)\phi_1(t,\tau)\,\mathrm{d}\mu=0. \tag{A13}$$

As u^ε is uniformly bounded in $\mathbb{L}^2(0,T;\mathbb{H}^1_0(\mathcal{O}))$, we deduce from Theorem 2 that u^ε converges weakly in $\mathbb{L}^2(0,T;\mathbb{H}^1_0(\mathcal{O}))$ to $u_0(t,x)$, therefore

$$\lim_{\varepsilon\searrow 0}\iint_{\mathbb{S}^2}\left\{\frac{u^\varepsilon(t,x)-u_0(t,x)}{\varepsilon}\right\}\frac{\partial \phi_1}{\partial \tau}(t,\tau)\phi_2(x,y)\,\mathrm{d}t\,\mathrm{d}x=\iiiint_{\mathbb{S}^4}u_1(t,x,\tau,y)\frac{\partial \phi_1}{\partial \tau}(t,\tau)\phi_2(x,y)\,\mathrm{d}\mu.$$

As $\varepsilon\searrow 0$, Equation (A13) becomes

$$\iiiint_{\mathbb{S}^4}\left[u_1(t,x,\tau,y)\frac{\partial \phi_1}{\partial \tau}\phi_2-\sigma(t,x,\tau,y)\left\{\frac{\partial u_0}{\partial x}(t,x)+\frac{\partial u_1}{\partial y}(t,x,\tau,y)\right\}\frac{\partial \phi_2}{\partial y}\phi_1\right]\mathrm{d}\mu=0. \tag{A14}$$

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By integrating by parts (A14) and using the separation of variables (A4) we obtain the local equation:

$$\frac{\mathrm{d}\chi}{\mathrm{d}\tau}(\tau, y) - \frac{\partial}{\partial y}\left\{\sigma(t, x, \tau, y)\left[1 + \frac{\partial \chi}{\partial y}(\tau, y)\right]\right\} = 0. \tag{A15}$$

Case 3: $\gamma > 2$

As ε goes to zero, it follows from the two-scale convergence method applied to (A10) that

$$\iiiint_{\mathbb{S}^4} u_1(t, x, \tau, y)\frac{\partial \phi_1}{\partial \tau}(t, \tau)\phi_2(x, y)\,\mathrm{d}\mu = 0 \quad \forall \phi_1 \in \mathbb{L}^2(0, T),\ \phi_2 \in H_0^1(\mathcal{O}, \mathbb{H}^1_{\#}(Y))$$

which is equivalent to

$$\iiiint_{\mathbb{S}^4} \frac{\partial u_1}{\partial \tau}(t, x, \tau, y)\phi_1(t, \tau)\phi_2(x, y)\,\mathrm{d}\mu = 0 \quad \forall \phi_1 \in \mathbb{L}^2(0, T),\ \phi_2 \in H_0^1(\mathcal{O}, \mathbb{H}^1_{\#}(Y)).$$

Therefore, $u_1(t, x, \tau, y)$ is constant in τ: $u_1(t, x, \tau, y) = u_1(t, x, y)$. The test function $\phi(t, x)$ has then to be constant in the τ direction. For $\phi(t, x) = \varepsilon^{\gamma-1}\phi_1(x, x/\varepsilon)\phi_2(t)$, Equation (A10) is replaced by

$$\begin{aligned}
&\varepsilon^{\gamma-2}\left[\iint_{\mathbb{S}^2}\sigma\left(t, x, \frac{t}{\varepsilon^\gamma}, \frac{x}{\varepsilon}\right)\frac{\partial u^\varepsilon}{\partial x}\frac{\partial \phi_2}{\partial y}\phi_1(t)\,\mathrm{d}t\,\mathrm{d}x - \iiint_{\mathbb{S}^3}\sigma(t, x, \tau, y)\left\{\frac{\partial u_0}{\partial x} + \frac{\partial u_1}{\partial y}\right\}\frac{\partial \phi_2}{\partial y}\phi_1(t)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y\right] \\
&\quad + \varepsilon^{\gamma-1}\left[\iint_{\mathbb{S}^2}\sigma\left(t, x, \frac{t}{\varepsilon^\gamma}, \frac{x}{\varepsilon}\right)\frac{\partial u^\varepsilon}{\partial x}\frac{\partial \phi_2}{\partial x}\phi_1(t)\,\mathrm{d}t\,\mathrm{d}x + \iint_{\mathbb{S}^2}\{u^\varepsilon(t, x) - u_0(t, x)\}\frac{\partial \phi_1}{\partial t}(t)\phi_2(x, y)\,\mathrm{d}t\,\mathrm{d}x\right] \\
&\quad - \varepsilon^{\gamma-1}\left[\iiint_{\mathbb{S}^3}\sigma(t, x, \tau, y)\left\{\frac{\partial u_0}{\partial x} + \frac{\partial u_1}{\partial y}\right\}\frac{\partial \phi_2}{\partial x}\phi_1(t)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y\right] = 0.
\end{aligned} \tag{A16}$$

As $\varepsilon \searrow 0$, the two-scale limit of the Equation (A16) is

$$\begin{aligned}
&\iiint_{\mathbb{S}^3}\left(\int_0^1 \sigma(t, x, \tau, y)\,\mathrm{d}\tau\right)\left[\frac{\partial u_0}{\partial x}(t, x) + \frac{\partial u_1}{\partial y}(t, x, y)\right]\frac{\partial \phi_2}{\partial y}(x, y)\phi_1(t)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y \\
&\quad - \iiint_{\mathbb{S}^3}\left\{\int_Y\left(\int_0^1 \sigma(t, x, y)\,\mathrm{d}\tau\right)\left[\frac{\partial u_0}{\partial x}(t, x) + \frac{\partial u_1}{\partial y}(t, x, y)\right]\frac{\partial \phi_2}{\partial y}(x, y)\phi_1(t)\,\mathrm{d}y\right\}\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y = 0.
\end{aligned} \tag{A17}$$

As

$$\int_Y\left(\int_0^1 \sigma(t, x, y)\,\mathrm{d}\tau\right)\left[\frac{\partial u_0}{\partial x}(t, x) + \frac{\partial u_1}{\partial y}(t, x, y)\right]\frac{\partial \phi_2}{\partial y}(x, y)\phi_1(t)\,\mathrm{d}y$$

is independent of the local variable y, then

$$\iiint_{\mathbb{S}^3}\left\{\int_Y\left(\int_0^1 \sigma(t, x, y)\,\mathrm{d}\tau\right)\left[\frac{\partial u_0}{\partial x}(t, x) + \frac{\partial u_1}{\partial y}(t, x, y)\right]\frac{\partial \phi_2}{\partial y}(x, y)\phi_1(t)\,\mathrm{d}y\right\}\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y = 0.$$

Consequently, (A17) reduces to

$$\iiint_{\mathbb{S}^3}\left(\int_0^1 \sigma(t, x, \tau, y)\,\mathrm{d}\tau\right)\left[\frac{\partial u_0}{\partial x}(t, x) + \frac{\partial u_1}{\partial y}(t, x, \tau, y)\right]\frac{\partial \phi_2}{\partial y}(x, y)\phi_1(t)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y = 0.$$

Similar to the separation of variables (A4), assuming $u_1(t, x, y) = (\partial u_0/\partial x)(t, x)\kappa(y)$ and integrating by parts with respect to y, the local equation results as

$$\frac{\partial}{\partial y}\left\{\left(\int_0^1 \sigma(t, x, \tau, y)\,\mathrm{d}\tau\right)\left[1 + \frac{\partial \kappa}{\partial y}(y)\right]\right\} = 0. \tag{A18}$$

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In light of the derivation of the local equation, it is worth noting that the global problem $\widetilde{\mathscr{P}b}$ in the case where $\gamma>2$ is

$$\widetilde{\mathscr{P}pb}=\begin{cases} u_0(t,x)\in\mathbb{L}^2(0,T;\mathbb{H}_0^1(\mathscr{O})),\\ \dfrac{\mathrm{d}u_0}{\mathrm{d}t}(t,x)-\dfrac{\partial}{\partial x}\left[\left\{\displaystyle\int_Y\left(\int_0^1\sigma(t,x,\tau,y)\mathrm{d}\tau\right)\left[1+\dfrac{\partial\kappa}{\partial y}(y)\right]\mathrm{d}y\right\}\dfrac{\partial u_0}{\partial x}(t,x)\right]=f(t,x)\quad\forall(t,x)\in\mathbb{S}^2,\\ u_0(t=0,x)=a(x)\quad\forall x\in\mathscr{O}.\end{cases}$$

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