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Calculation of Uncertainty Propagation using Adjoint Equations

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Dedicated to Professor Mutsuto Kawahara on the occasion of his 60th birthday

The uncertainty of flow parameters depending on the error of input data (initial, boundary conditions, coefficients) may be efficiently calculated using adjoint equations. This approach is extremely effective for uncertainty estimation at certain checkpoints because it requires only a single (adjoint) system of equations to be solved in addition to the system describing the flow-field. The fields of adjoint "temperature", adjoint "density", etc. are then used to calculate the transfer of uncertainty from all input data.

Keywords: Uncertainty; Adjoint equations; Propagation; Density

INTRODUCTION

The uncertainty of flow-field parameters depending on the error of initial conditions, boundary conditions, and coefficients may be calculated by a number of approaches such as Monte-Carlo methods or sensitivity equations. Unfortunately, the algorithms that estimate both the flow parameters and their uncertainty are very rare because they usually require an extensive amount of computation time. For example, the dispersion of a result ε may be calculated from the input data f_i dispersion using the sensitivity coefficients $\partial\varepsilon/\partial f_i$ (Putko *et al.*, 2001).

$$\sigma_\varepsilon^2 = \sum_{i=1}^{N}\left(\frac{\partial\varepsilon}{\partial f_i}\sigma_{f_i}\right)^2. \qquad (1)$$

The time required for $\partial\varepsilon/\partial f_i$ calculation using either sensitivity equations or the direct numerical differentiation of ε is proportional to the time required for the ε calculation multiplied by the number N of input parameters. The total time required for $\partial\varepsilon/\partial f_i$ calculation is practically unacceptable if the computational effort of ε calculation is high and N is large.

The present paper addresses the issue of estimation of uncertainty using coefficients $\partial\varepsilon/\partial f_i$ calculated via adjoint equations. If the uncertainty is estimated with a single checkpoint, this approach requires minimal computational resources since only one adjoint system should be solved independently on number N. The time for $\partial\varepsilon/\partial f_i$ calculation is approximately equal to double the time required for ε computation in this event.

The fields of adjoint "temperature", adjoint "density", etc. depend on the flow-field, estimated parameters and checkpoint location and do not depend on the set of input data, which have an uncertainty. So, they are universal and allow for the calculation of uncertainty caused by any parameter of the system of equations.

The adjoint approach is illustrated herein for the parabolized Navier–Stokes. See also Putko *et al.* (2001); Alekseev (2001); Alekseev and Navon (2001; 2002).

FLOW PARAMETER UNCERTAINTY ESTIMATION

We consider the uncertainty estimation in supersonic viscous flow, Fig. 1. The flow parameters are calculated by the finite-difference approximation of parabolized

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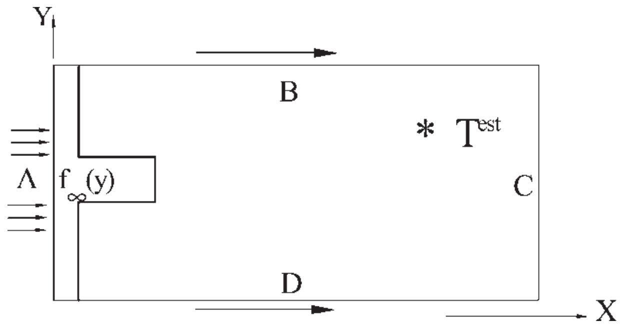

FIGURE 1

Navier–Stokes (Alekseev, 2001; Alekseev and Navon, 2001). The march along X coordinate was used.

$$\frac{\partial(\rho U)}{\partial X}+\frac{\partial(\rho V)}{\partial Y}=0 \tag{2}$$

$$U\frac{\partial U}{\partial X}+V\frac{\partial U}{\partial Y}+\frac{1}{\rho}\frac{\partial P}{\partial X}=\frac{1}{\mathrm{Re}\,\rho}\frac{\partial^2 U}{\partial Y^2} \tag{3}$$

$$U\frac{\partial V}{\partial X}+V\frac{\partial V}{\partial Y}+\frac{1}{\rho}\frac{\partial P}{\partial Y}=\frac{4}{3\rho\,\mathrm{Re}}\frac{\partial^2 V}{\partial Y^2} \tag{4}$$

$$U\frac{\partial e}{\partial X}+V\frac{\partial e}{\partial Y}+(\kappa-1)e\left(\frac{\partial U}{\partial X}+\frac{\partial V}{\partial Y}\right)=\frac{1}{\rho}\left(\frac{\kappa}{\mathrm{Re\,Pr}}\frac{\partial^2 e}{\partial Y^2}+\frac{4}{3\,\mathrm{Re}}\left(\frac{\partial U}{\partial Y}\right)^2\right) \tag{5}$$

$$P=\rho\mathrm{R}T;\quad e=C_vT;\quad (X,Y)\in\Omega=(0<X<X_{\max};\ 0<Y<1).$$

The entrance boundary ($A(X=0)$, Fig. 1) conditions follow:

$$\begin{aligned}&e(0,Y)=e_\infty(Y);\quad \rho(0,Y)=\rho_\infty(Y));\\&U(0,Y)=U_\infty(Y);\quad V(0,Y)=V_\infty(Y);\end{aligned} \tag{6}$$

while outflow conditions $\partial f/\partial Y=0$ are used on B, $D(Y=0, Y=1)$.

Let the inflow parameters contain the uncertainty. We assume the discrete analogues of these parameters contain a normally distributed error ($\sigma_\rho, \sigma_U, \sigma_V, \sigma_e$).

Let us seek for the total flow-field and the accuracy of a certain parameter (let it be the temperature) at some check point $T(t_{\mathrm{est}}, x_{\mathrm{est}})$, more precisely: a dependence of the temperature standard deviation from the input data deviations $\sigma_T=f(\sigma_\rho,\sigma_U,\sigma_V\cdots)$.

Let us denote $T(X^{\mathrm{est}}, Y^{\mathrm{est}})$ as $\varepsilon(f_\infty(Y),\mathrm{Re})$. If the estimated parameter is located on the outflow boundary we may express it as

$$\varepsilon(f_\infty(Y))=\int T(X_{\max},Y)\delta(Y-Y^{\mathrm{est}})\mathrm{d}y \tag{7}$$

If $T(X^{\mathrm{est}}, Y^{\mathrm{est}})$ is located within the field we write

$$\varepsilon(f_\infty(Y))=\int_\Omega T(X,Y)\,\delta(Y-Y^{\mathrm{est}})\delta(X-X^{\mathrm{est}})\mathrm{d}x\mathrm{d}y. \tag{8}$$

The input data dispersion is transformed to the result dispersion by gradients (Putko *et al.*, 2001), in our case

$$\sigma_\varepsilon^2=\sum_{i=1}^{NY}\left(\frac{\partial\varepsilon}{\partial T_\infty^i}\sigma_{T_i}\right)^2+\sum_{i=1}^{NY}\left(\frac{\partial\varepsilon}{\partial\rho_\infty^i}\sigma_{\rho_i}\right)^2+\sum_{i=1}^{NY}\left(\frac{\partial\varepsilon}{\partial U_\infty^i}\sigma_{U_i}\right)^2+\sum_{i=1}^{NY}\left(\frac{\partial\varepsilon}{\partial V_\infty^i}\sigma_{V_i}\right)^2+\left(\frac{\partial\varepsilon}{\partial(1/\mathrm{Re})}\sigma_{(1/\mathrm{Re})}\right)^2. \tag{9}$$

The most efficient way for gradient calculation is based on adjoint equations. For these equations' derivation we introduce the Lagrangian $L(f_\infty(Y),\Psi)$, composed of the estimated value and the weak statement of problems (2–5).

$$\begin{aligned}L(f_\infty(Y),\mathrm{Re})&=\varepsilon(f_\infty(Y))=\int_\Omega T(X,Y)\delta(X-X^{\mathrm{est}})\delta(Y-Y^{\mathrm{est}})\mathrm{d}X\,\mathrm{d}Y\\&+\iint_{XY}\left(\frac{\partial(\rho U)}{\partial X}+\frac{\partial(\rho V)}{\partial Y}\right)\Psi_\rho(X,Y)\mathrm{d}X\,\mathrm{d}Y\\&+\iint_{XY}\left(U\frac{\partial U}{\partial X}+V\frac{\partial U}{\partial Y}+\frac{1}{\rho}\frac{\partial P}{\partial X}-\frac{1}{\mathrm{Re}\,\rho}\frac{\partial^2 U}{\partial Y^2}\right)\Psi_U(X,Y)\mathrm{d}X\,\mathrm{d}Y\\&+\iint_{XY}\left(U\frac{\partial V}{\partial X}+V\frac{\partial V}{\partial Y}+\frac{1}{\rho}\frac{\partial P}{\partial Y}-\frac{4}{3\mathrm{Re}\,\rho}\frac{\partial^2 V}{\partial Y^2}\right)\Psi_V(X,Y)\mathrm{d}X\,\mathrm{d}Y\\&+\iint_{XY}\left(U\frac{\partial e}{\partial X}+V\frac{\partial e}{\partial Y}+(\kappa-1)e\left(\frac{\partial U}{\partial X}+\frac{\partial V}{\partial Y}\right)\right)\Psi_e(X,Y)\mathrm{d}X\,\mathrm{d}Y\\&+\iint_{XY}\left(-\frac{\kappa}{\rho\mathrm{Re\,Pr}}\frac{\partial^2 e}{\partial Y^2}-\frac{4}{3\mathrm{Re}\,\rho}\frac{\partial U}{\partial Y}\right)^2\Big)\Psi_e(X,Y)\mathrm{d}X\,\mathrm{d}Y.\end{aligned} \tag{10}$$

Consider the influence of inflow data variation $\Delta f_\infty(X)$ and coefficient variation $\Delta(1/\mathrm{Re})$. By subtracting the undisturbed solution we get the linear tangent model:

$$U\frac{\partial(\Delta\rho)}{\partial X}+\rho\frac{\partial(\Delta U)}{\partial X}+\rho\frac{\partial(\Delta V)}{\partial Y}+V\frac{\partial(\Delta\rho)}{\partial Y}+\Delta\rho\frac{\partial U}{\partial X}+\Delta U\frac{\partial\rho}{\partial X}+\Delta\rho\frac{\partial V}{\partial Y}+\Delta V\frac{\partial\rho}{\partial Y}=0, \tag{11}$$

$$U\frac{\partial \Delta U}{\partial X} + \Delta U\frac{\partial U}{\partial X} + \Delta V\frac{\partial U}{\partial Y} + V\frac{\partial \Delta U}{\partial Y} - \frac{1}{\rho \,\mathrm{Re}}\frac{\partial^2 \Delta U}{\partial Y^2} + \frac{\Delta\rho}{\rho^2\,\mathrm{Re}}\frac{\partial^2 U}{\partial Y^2} - \frac{\Delta\rho}{\rho^2}\frac{\partial P}{\partial X} + \frac{(\kappa-1)}{\rho}\left(\Delta\rho\frac{\partial e}{\partial X} + e\frac{\partial\Delta\rho}{\partial X} + \rho\frac{\partial\Delta e}{\partial X} + \Delta e\frac{\partial\rho}{\partial X}\right) - \frac{1}{\rho}\frac{\partial^2 U}{\partial Y^2}\Delta(1/\mathrm{Re}) = 0 \quad (12)$$

$$U\frac{\partial \Delta V}{\partial X} + \Delta U\frac{\partial V}{\partial X} + \Delta V\frac{\partial V}{\partial Y} + V\frac{\partial \Delta V}{\partial Y} - \frac{4}{3\,\rho\,\mathrm{Re}}\frac{\partial^2 \Delta V}{\partial Y^2} + \frac{4\Delta\rho}{3\rho^2\,\mathrm{Re}}\frac{\partial^2 V}{\partial Y^2} - \frac{\Delta\rho}{\rho^2}\frac{\partial P}{\partial Y} + \frac{(\kappa-1)}{\rho}\left(\Delta\rho\frac{\partial e}{\partial Y} + e\frac{\partial\Delta\rho}{\partial Y} + \rho\frac{\partial\Delta e}{\partial Y} + \Delta e\frac{\partial\rho}{\partial Y}\right) - \frac{4}{3\,\rho}\frac{\partial^2 V}{\partial Y^2}\Delta(1/\mathrm{Re}) = 0 \quad (13)$$

$$U\frac{\partial \Delta e}{\partial X} + \Delta U\frac{\partial e}{\partial X} + \Delta V\frac{\partial e}{\partial Y} + V\frac{\partial \Delta e}{\partial Y} + (\kappa-1)\Delta e\left(\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y}\right) + (\kappa-1)e\left(\frac{\partial \Delta U}{\partial X} + \frac{\partial \Delta V}{\partial Y}\right) - \frac{1}{\rho}\left(\frac{\kappa}{\mathrm{Re\,Pr}}\frac{\partial^2\Delta e}{\partial Y^2} - \frac{\kappa}{\mathrm{Re\,Pr}}\frac{\Delta\rho}{\rho}\frac{\partial^2 e}{\partial Y^2}\right) - \frac{1}{\rho}\left(\frac{8}{3\,\mathrm{Re}}\left(\frac{\partial\Delta U}{\partial Y}\right)\left(\frac{\partial U}{\partial Y}\right) - \frac{\Delta\rho}{\rho}\frac{4}{3\,\mathrm{Re}}\left(\frac{\partial U}{\partial Y}\right)^2\right) - \frac{1}{\rho}\left(\frac{\kappa}{\mathrm{Pr}}\frac{\partial^2 e}{\partial Y^2} + \frac{4}{3}\left(\frac{\partial U}{\partial Y}\right)^2\right)\Delta(1/\mathrm{Re}) = 0. \quad (14)$$

On the boundaries the variations of $\Delta\rho, \Delta U, \Delta V, \Delta e$ should satisfy inflow boundary conditions:

$$\Delta e(0,Y) = \Delta e_\infty(Y),\ \Delta\rho(0,Y) = \Delta\rho_\infty(Y), \quad (15)$$
$$\Delta\rho(0,Y) = \Delta\rho_\infty(Y),\ \Delta V(0,Y) = \Delta V_\infty(Y)$$

and lateral boundary conditions:

$$B(Y=1):\ \Delta e(X,1) = 0,\ \Delta\rho(X,1) = 0,\ \Delta U(X,1) = 0,\ \Delta V(X,1) = 0.$$

$$D(Y=0):\ \Delta e(X,0) = 0,\ \Delta\rho(X,0) = 0,\ \Delta U(X,0) = 0,\ \Delta V(X,0) = 0.$$

We use equations (11–15) for Lagrangian (10) variation statement.

$$\Delta\varepsilon(Q_w(t)) = \int_\Omega \Delta T\,\delta(t - t_{\mathrm{est}})\delta(x - x_{\mathrm{est}})\mathrm{d}t\mathrm{d}x + \ldots \quad (16)$$

Integrating the Lagrangian variation (16) by parts taking into account Eqs. (11–15) allows estimation of the variation of the target parameter as a function of the disturbed parameters (17).

$$\Delta L(f_\infty(Y), f, \Psi) = \Delta\varepsilon(f_\infty(Y)) = \int_Y ((\Psi_e U + (\kappa-1)\Psi_U)\Delta e_\infty(Y))|_{X=0}\,\mathrm{d}Y + \int_Y ((\Psi_\rho U + (\kappa-1)\Psi_U e/\rho)\Delta\rho_\infty(Y))|_{X=0}\,\mathrm{d}Y + \int_Y ((\Psi_U U + \rho\Psi_\rho + (\kappa-1)\Psi_e e)\Delta U_\infty(Y))|_{X=0}\,\mathrm{d}Y + \int_Y (\Psi_V U \Delta V_\infty(Y))|_{X=0}\mathrm{d}Y - \Delta(1/\mathrm{Re}) \times \int_\Omega \left(\frac{1}{\rho}\frac{\partial^2 U}{\partial Y^2}\Psi_U + \frac{4}{3\,\rho}\frac{\partial^2 V}{\partial Y^2}\Psi_V + \frac{\kappa}{\rho\,\mathrm{Pr}}\frac{\partial^2 e}{\partial Y^2}\Psi_e + \frac{4}{3\,\rho}\left(\frac{\partial U}{\partial Y}\right)^2\Psi_e\right)\mathrm{d}X\,\mathrm{d}Y. \quad (17)$$

Equation (17) is valid if the remaining terms of $\Delta L(f_\infty(Y) f, \Psi)$ equal zero, i.e. on the solution of the adjoint problem provided by the following equations (18–22).

ADJOINT PROBLEM

$$U\frac{\partial\Psi_\rho}{\partial X} + V\frac{\partial\Psi_\rho}{\partial Y} + (\kappa-1)\frac{\partial(\Psi_V e/\rho)}{\partial Y} + (\kappa-1)\frac{\partial(\Psi_U e/\rho)}{\partial X} - \frac{\kappa-1}{\rho}\left(\frac{\partial e}{\partial Y}\Psi_V + \frac{\partial e}{\partial X}\Psi_U\right) + \left(\frac{1}{\rho^2}\frac{\partial P}{\partial X} - \frac{1}{\rho^2\,\mathrm{Re}}\frac{\partial^2 U}{\partial Y^2}\right)\Psi_U + \frac{1}{\rho^2}\left(\frac{\partial P}{\partial Y} - \frac{4}{3\,\mathrm{Re}}\frac{\partial^2 V}{\partial Y^2}\right)\Psi_V - \frac{1}{\rho^2}\left(\frac{\kappa}{\mathrm{Re\,Pr}}\frac{\partial^2 e}{\partial Y^2} + \frac{4}{3\,\mathrm{Re}}\left(\frac{\partial U}{\partial Y}\right)^2\right)\Psi_e = 0 \quad (18)$$

$$U\frac{\partial\Psi_U}{\partial X}+\frac{\partial(\Psi_U V)}{\partial Y}+\rho\frac{\partial\Psi_\rho}{\partial X}-\left(\frac{\partial V}{\partial X}\Psi_V+\frac{\partial e}{\partial X}\Psi_e\right)$$
$$+\frac{\partial}{\partial X}\left(\frac{P}{\rho}\Psi_e\right)+\frac{\partial^2}{\partial Y^2}\left(\frac{1}{\rho \mathrm{Re}}\Psi_U\right)-\frac{\partial}{\partial Y}\left(\frac{8}{3\mathrm{Re}}\frac{\partial U}{\partial Y}\Psi_e\right)=0 \quad (19)$$

$$\frac{\partial(U\Psi_V)}{\partial X}+V\frac{\partial\Psi_V}{\partial Y}-\left(\frac{\partial U}{\partial Y}\Psi_U+\frac{\partial e}{\partial Y}\Psi_e\right)$$
$$+\rho\frac{\partial\Psi_\rho}{\partial Y}+\frac{\partial}{\partial Y}\left(\frac{P}{\rho}\Psi_e\right)+\frac{4}{3\mathrm{Re}}\frac{\partial^2}{\partial Y^2}\left(\frac{\Psi_V}{\rho}\right)=0 \quad (20)$$

$$\frac{\partial(U\Psi_e)}{\partial X}+\frac{\partial(V\Psi_e)}{\partial Y}-\frac{\kappa-1}{\rho}\left(\frac{\partial\rho}{\partial Y}\Psi_V+\frac{\partial\rho}{\partial X}\Psi_U\right)$$
$$-(\kappa-1)\left(\frac{\partial U}{\partial X}+\frac{\partial V}{\partial Y}\right)\Psi_e+(\kappa-1)\frac{\partial\Psi_V}{\partial Y}$$
$$+(\kappa-1)\frac{\partial\Psi_U}{\partial X}+\frac{\kappa}{\mathrm{RePr}}\frac{\partial^2}{\partial Y^2}\left(\frac{\Psi_e}{\rho}\right)$$
$$-\delta(X-X^{\mathrm{est}})\delta(Y-Y^{\mathrm{est}})=0. \quad (21)$$

The source term in Eq. (21) describing Ψ_e corresponds to the checkpoint location within the flow-field.

Initial conditions are

$$C(X=X_{\max}):\Psi_{\rho,U,V}|^{X=X_{\max}}=0;$$
$$U\Psi_e+(\kappa-1)\Psi_U+\delta(Y-Y^{\mathrm{est}})=0. \quad (22)$$

The expression for Ψ_e in Eq. (22) corresponds to the checkpoint location on the inflow boundary $X_{\max}$.

Boundary conditions for B and D are:

$$B,D(Y=0;\ Y=1):\frac{\partial\Psi_f}{\partial Y}=0. \quad (23)$$

The statement (18–23) differs from the adjoint equations used in inverse CFD problems by the form of the target functional and, respectively, by the source term form in Eqs. (21) and (22). The adjoint problem is solved in the reverse direction along X. Its statement is determined by the forward problem, checkpoint position, and the choice of the estimated parameter. The adjoint problem does not depend on the choice of parameters containing the uncertainty. So, the same field of adjoint parameters may be used for the calculation of uncertainty propagation from any parameters (initial, boundary conditions, coefficients, sources). The gradients used for the uncertainty propagation (see, Eq. (9)) assume the following form:

$$\partial\varepsilon/\partial e_\infty(Y)=\Psi_e U+(\kappa-1)\Psi_U,$$
$$\partial\varepsilon/\partial\rho_\infty(Y)=\Psi_\rho U+(\kappa-1)\Psi_U e/\rho$$
$$\partial\varepsilon/\partial U_\infty(Y)=\Psi_U U+\rho\Psi_\rho+(\kappa-1)\Psi_e e,$$
$$\partial\varepsilon/\partial V_\infty(Y)=\Psi_V U$$

$$\nabla\varepsilon_{1/\mathrm{Re}}=-\int_\Omega\left(\frac{1}{\rho}\frac{\partial^2 U}{\partial Y^2}\Psi_U+\frac{4}{3\rho}\frac{\partial^2 V}{\partial Y^2}\Psi_V+\frac{\kappa}{\rho\mathrm{Pr}}\frac{\partial^2 e}{\partial Y^2}\Psi_e+\frac{4}{3\rho}\left(\frac{\partial U}{\partial Y}\right)^2\Psi_e\right)\mathrm{d}X\mathrm{d}Y \quad (24)$$

The calculation of the gradient implies the consequent solution of the direct and adjoint problems. So, the time required for uncertainty calculation of the single parameter in the single checkpoint equals approximately twice that required for the flow-field calculation. The uncertainty estimation of every additional parameter needs the solution of additional adjoint equation.

TEST RESULTS

The singular source in Eq. (21) is integrated over the cell and thus transformed to the finite source term $\delta_{ij}/(\Delta X\Delta Y)$, if the checkpoint is located within flow-field, and to $\delta_{ij}/\Delta Y$ if the checkpoint is on the boundary (22), where δ_{ij} is the unit matrix.

The uncertainty propagation is calculated by the adjoint equations and Monte-Carlo method (averaged over 100 trials) for the comparison. The inflow parameters contain the normally distributed error with the standard deviation in the range of 0.01–0.1. The standard deviations of temperature at the middle point ($N=50$) on the outflow boundary calculated by both methods are presented in Table I for the case of uniform flow.

Adjoint "density" field is presented in Fig. 2, adjoint "temperature" is presented in Fig. 3.

The flow-field parameters and their gradients form the sources and coefficients of the adjoint equations. So, the following tests are conducted for non-uniform flow corresponding to an underexpanded jet with the temperature ratio $Tj/T=3$ (density isolines are provided in Fig. 4). The coincidence of adjoint and Monte-Carlo approaches is of the same quality. The temperature

TABLE I Uniform flow-field ($N_{\mathrm{est}}=50$)

σ_{f_∞}	σ_T, adjoint approach	σ_T, averaged over 100 runs
0.01	0.0024	0.002613
0.05	0.012	0.0144
0.1	0.024	0.03

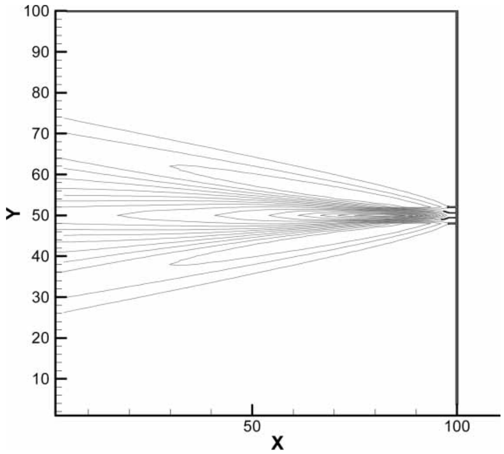

FIGURE 2

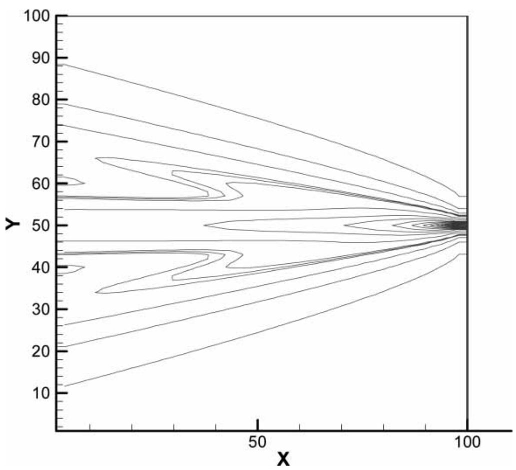

FIGURE 3

TABLE II Underexpanded jet ($N_{\text{est}} = 50$)

σ_{f_∞}	σ_T, adjoint approach	σ_T, averaged over 100 runs
0.01	0.0029	0.0029
0.05	0.0145	0.0152
0.1	0.029	0.03

TABLE III Underexpanded jet, ($N_{\text{est}} = 20$)

σ_{f_∞}	σ_T, adjoint approach	σ_T averaged over 100 runs
0.01	0.00284	0.00304
0.05	0.0142	0.0148
0.1	0.0284	0.034

uncertainty estimations are presented in Tables II and III for different checkpoint locations ($N_{\text{est}} = 50$ and $N_{\text{est}} = 20$). By comparing Figs. 5 and 6 we may see the difference of regions from which the main part of uncertainty is propagated.

Figures 2, 3, 5 and 6 correspond to the checkpoint located on the boundary, (Eq. (22)); Fig. 7 corresponds to the checkpoint within the flow-field (Eq. (21)). Generally, the results of both approaches (namely Monte-Carlo and adjoint) correlate, while the required computer time differs by two orders of magnitude for these tests, respectively.

The computation of the single parameter (T) at the single check point requires the calculation of the flow-field and the adjoint field. The estimation of another parameter (or the same one but at another check point) requires calculation of the new adjoint field with the same flow-field.

DISCUSSION

Let's compare the different approaches for the uncertainty calculation from the viewpoint of necessary computer resources.

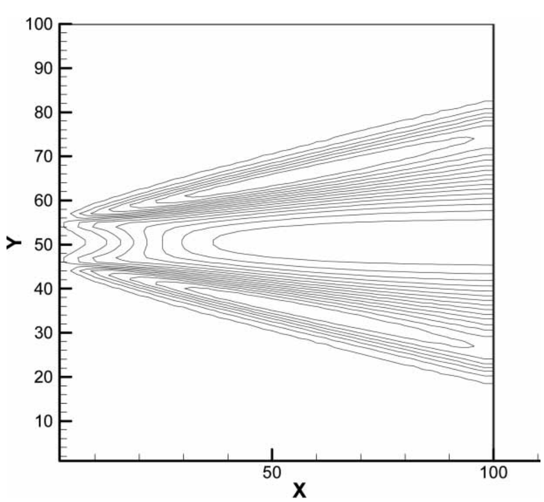

FIGURE 4

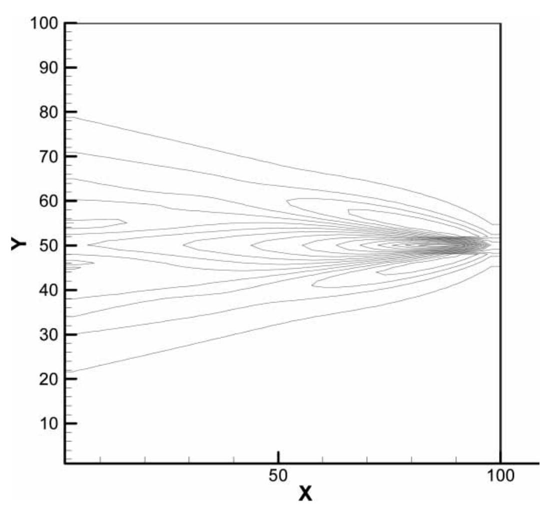

FIGURE 5

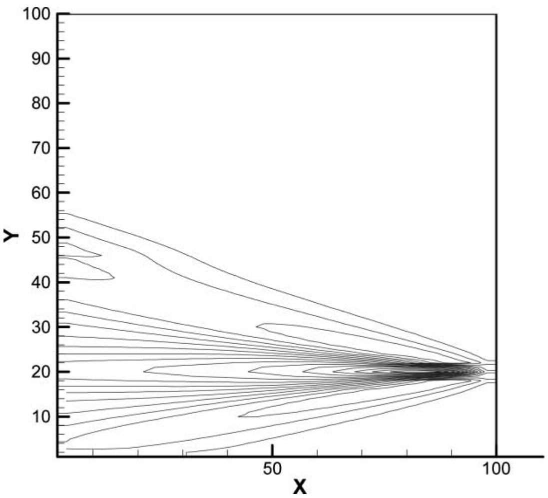

FIGURE 6

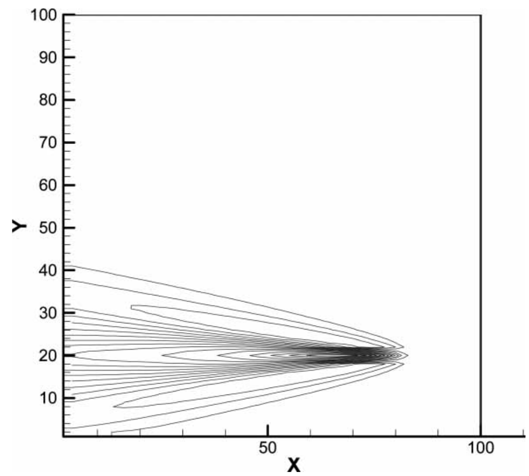

FIGURE 7

The temperature dispersion for total flow-field may be calculated by the sensitivity equations written for values $S_k(X,Y,X_k) = \partial T(X,Y)/\partial f^i_\infty(X_k)$. This approach implies the solution of the system of equations of higher dimension (by the dimension of the space of control parameters) in comparison with the system (2–5). For the problem under consideration it may mean $4N_y$ calculations of Eqs. (2–5). The adjoint equations require $N_xN_y + 1$ calculations of Eqs. (2–5) for the temperature dispersion in the total flow-field. Thus, the sensitivity equations are less expensive if the total field of dispersion is calculated. Nevertheless, the adjoint equations are far more efficient from a computational time viewpoint for a relatively small set of estimated parameters.

Another approach to the uncertainty estimation based on the adjoint equations of the second order is described by Alekseev and Navon (2002). The target functional for this approach has a form:

$$\varepsilon(\delta f_\infty(Y)) = \int_X \left(T^{\text{exact}}_{Y_{\text{est}}}(X) - T^{\text{error}}_{Y_{\text{est}}}(X) \right)^2 \mathrm{d}X.$$

The second-order adjoint approach uses calculation of Hessian (or part of its spectrum) and requires a computational time proportional to the number of parameters containing error, which is for the considered problem of the order of N_y. In general, the second-order approach seems to be most suitable for the calculation of the uncertainty in the inverse CFD problem.

Monte-Carlo methods are also expensive from the computational time standpoint, although they may be implemented much more simply since they do not need the solution of any auxiliary problem.

In general, the considered adjoint method is the most efficient from a computational time viewpoint if we calculate the uncertainty of parameters at a small set of checkpoints.

The adjoint approach implies the simultaneous solution of the set of adjoint equations having the slightest differences. So, the computation of the uncertainty may easily be parallelized.

CONCLUSION

The uncertainty of the flow parameter in the checkpoint from input data error may be calculated using adjoint equations under a total computer time consumption corresponding to the double calculation of the flow-field.

The calculation of the uncertainty of n parameters needs the calculation of $n + 1$ fields (flow-field + n adjoint fields).

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