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A Global Uniformly Convergent Finite Element Method for a Quasi-Linear Singularly Perturbed Elliptic Problem

JICHUN LI
Texas Institute for Computational and Applied Mathematics
TAY 2.400, University of Texas at Austin
Austin, TX 78712, U.S.A.
jli@ticam.utexas.edu

I. M. NAVON*
Department of Mathematics
and Supercomputer Computations Research Institute
Florida State University, Tallahassee, FL 32306, U.S.A.
navon@scri.fsu.edu

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Abstract—In this paper, we construct a bilinear finite element method based on a special piecewise uniform mesh for solving a quasi-linear singularly perturbed elliptic problem in two space dimensions. A quasi-optimal global uniform convergence rate $O(N_x^{-2}\ln^2 N_x + N_y^{-2}\ln^2 N_y)$ was obtained, which is independent of the perturbation parameter. Here N_x and N_y are the number of elements in the x-and y-directions, respectively.

Keywords—Finite element method, Singular perturbation, Uniformly convergent.

1. INTRODUCTION

Singularly perturbed problems (SPP) appear in many branches of applied mathematics, for example, in fluid mechanics [1], chemical kinetics [2], biochemical kinetics [3, Chapter 10], and system control [2,4,5], etc. Such problems arise naturally when there are sudden transitions from certain physical characteristics to others. These transitions can occur either inside a very thin layer near the boundary or inside the problem domain. Such a thin layer is called the boundary layer or internal layer. These layers make the problem very difficult to solve both analytically [2,6,7] and numerically [8–11].

While a sizable amount of work has been carried out using methods, such as *finite difference methods* [8,12], *spectral methods* [13–15], *finite volume methods* [9,16], and *finite element methods* (FEM) [10,17–19], to name but a few, a large number of unsolved problems still remain to be

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addressed. For example, by using the standard bilinear FEM to solve the simple reaction-diffusion problem

$$-\varepsilon^2 \triangle u + u = f(x,y), \qquad \text{in } \Omega \subseteq R^2, \quad u|_{\partial\Omega} = 0,$$

where $0 < \varepsilon \ll 1$ is a perturbation parameter, we have the following global error estimates:

$$\|u - u_h\|_\varepsilon \leq C\,(\varepsilon + h)\,h\|u\|_{H^2(\Omega)},$$

where $\|u\|_\varepsilon = (\varepsilon^2\|\nabla u\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)})^{1/2}$. By simple calculation, we have $\|u\|_{H^2(\Omega)} \leq C\varepsilon^{-2}\|f\|_{L^2(\Omega)}$ [20, Lemma 2.1], hence, to ensure global convergence, the mesh size h must be in the order of $o(\varepsilon)$. However, ε can often be as small as 10^{-8}, in which case, h must be of order $o(10^{-8})$, which is very impractical. Hence, much research work focused only on local error estimates [21,22]. However, the global uniformly convergent (GUC) method is still very fascinating, since the error estimate is independent of the perturbation parameter ε. In the following, we will focus on GUC schemes obtained by FEM. As for other discretization methods, details can be found in the above-mentioned references.

It is well known that a global uniform convergence can be achieved by the exponential fitted FEM [18,23]. However, they are complicated to use and have a very low convergence rate, e.g., which is $\|u - u_h\|_\varepsilon \leq ch^{1/2}$ [18,20] for the convection-diffusion model

$$-\varepsilon \triangle u + b \cdot \nabla u + cu = f, \qquad \text{in } \Omega \subseteq R^2, \quad u|_{\partial\Omega} = 0.$$

Another type of uniform convergence was achieved for some very simple models by using hp FEM [24]. This method is very complicated and it is now only applied for the one space dimensional reaction-diffusion model [24]. Recently, almost optimal uniform convergence results were achieved by FEM on some specially designed piecewise uniform meshes [25–27], a method which was introduced by Shishkin [28]. This type of mesh specifies a fine uniform mesh inside part, but not all of the boundary layer and coarse uniform mesh elsewhere *a priori*, yet it still yields global convergence that is independent of ε. Such a mesh is very easy to implement, but the aforementioned studies were restricted only to one space dimensional problems until 1996 as evidenced by [24, p. 717]: "*These meshes work well for a wide range of one-dimensional problems. In two or more dimensions, however, the analysis of finite element methods on Shishkin meshes is an open question.*" Also, [10, p. 278]: "*Finite element methods that use Shishkin meshes in two or more dimensions have not been explored in the literature.*" To our best knowledge, the only available analysis for two space dimensional problems by using FEM on such piecewise uniform meshes are [29,30] for the convection-diffusion type problem, [31] for the reaction-diffusion problem and [32] for the anisotropic model problem.

In this paper, we will consider the following quasi-linear singularly perturbed elliptic problem:

$$\varepsilon^2 \triangle u = F(u,x,y), \qquad \text{in } \Omega = (0,1) \times (0,1), \tag{1.1}$$

$$u = 0, \qquad \text{on } \partial\Omega. \tag{1.2}$$

This problem was once discussed by Boglaev [33], where a nonlinear finite difference scheme was constructed. But the uniform convergence rate at the nodal points is only $O(N^{-1/2})$, where N is the total number of grid points. A similar problem was discussed in [20, p. 82], where only abstract error estimates were presented. Hereby by using the techniques developed in [29,31,32], we construct a bilinear FEM for solving the problem (1.1),(1.2) on a piecewise uniform mesh, where the quasi-optimal global uniform convergence

$$\|u - u_h\|_{L^2(\Omega)} \leq C\left(N_x^{-2}\ln^2 N_x + N_y^{-2}\ln^2 N_y\right)$$

is obtained. Here u_h denotes the FEM solution of (3.2), and N_x and N_y are the number of elements in the x- and y-directions, respectively.

The organization of this paper is as follows. In Section 2, we present the asymptotic expansion and the derivative estimates for the solution of (1.1),(1.2). Then a piecewise uniform mesh and a bilinear finite element scheme are constructed in Section 3. The quasi-optimal uniform convergence is proved in Section 4. Finally, an iterative scheme for solving the resulting nonlinear finite element system equations is presented in Section 5.

Throughout the paper, C will denote a generic positive constant, which is independent of the mesh size and the perturbation parameter ε. Also, we use the notation $\|\cdot\|_{\infty,\tau}$ for the L^∞ norm on τ, and $\|\cdot\|$ for the L^2 norm on Ω.

2. THE ASYMPTOTIC EXPANSION AND DERIVATIVE ESTIMATES

The asymptotic expansion for the problem (1.1),(1.2) is based on the work of Denisov [34], where F was assumed to be dependent on ε and in the form of $A(u^2 + pu + q)$. For simplicity, hereby we assume F is independent of ε. Also, the coefficients A, p, and q depend on x and y, and are assumed to be sufficiently smooth. Also, the following conditions are assumed [34, p. 1342].

(A1) The equation $F(u, x, y) = 0$ has a solution $u = \overline{u}_0(x, y)$ in $\overline{\Omega}$.

(A2) The derivative of F satisfies $m_2 \geq F_u(u(x, y), x, y) \geq m_1 > 0$ in $\overline{\Omega}$.

Under the above assumption, Denisov obtained the following.

LEMMA 2.1. *(See [34, p. 1349].) Denote the n^{th} order asymptotic expansion*

$$U_n(x, y, \varepsilon) = \sum_{k=0}^{n} \varepsilon^k \left(\overline{u}_k + \Pi_k^{(1)} + \cdots + \Pi_k^{(4)} + P_k^{(1)} + \cdots + P_k^{(4)}\right). \tag{2.1}$$

Then we have

$$\max_{\overline{\Omega}} |u(x, y, \varepsilon) - U_n(x, y, \varepsilon)| = O\left(\varepsilon^{n+1}\right), \qquad \text{as } \varepsilon \to 0, \tag{2.2}$$

where

$$\begin{aligned}
&\Pi^{(1)} = \Pi^{(1)}(x, \eta), && \eta = \frac{y}{\varepsilon}, && \Pi^{(2)} = \Pi^{(2)}(\xi, y), && \xi = \frac{x}{\varepsilon}, \\
&\Pi^{(3)} = \Pi^{(3)}(x, \eta_*), && \eta_* = \frac{1-y}{\varepsilon}, && \Pi^{(4)} = \Pi^{(4)}(\xi_*, y), && \xi_* = \frac{1-x}{\varepsilon}, \\
&P^{(1)} = P^{(1)}(\xi, \eta), && P^{(2)} = P^{(2)}(\xi, \eta_*), && P^{(3)} = P^{(3)}(\xi_*, \eta_*), && P^{(4)} = P^{(4)}(\xi_*, \eta).
\end{aligned}$$

The additional details for each term are presented in the following.

$\overline{u}_k(x, y)$ is the regular part of the asymptotic form. It satisfies the following equations:

$$\begin{aligned}
&F(\overline{u}_0(x, y), x, y) = 0, \\
&F_u(\overline{u}_0(x, y), x, y)\, \overline{u}_k(x, y) = \overline{G}_k(\overline{u}_0(x, y), \ldots, \overline{u}_{k-1}(x, y)), \qquad k = 1, 2, \ldots, n,
\end{aligned}$$

where the functions G_k depend on $\overline{u}_j(x, y)$, where $j < k$.

The functions $\Pi_k^{(i)}$ eliminate the discrepancies on the four sides of Ω. $\Pi_0^{(1)}$ satisfies:

$$\begin{aligned}
&\frac{\partial^2 \Pi_0^{(1)}}{\partial \eta^2} = F\left(\overline{u}_0(x, 0) + \Pi_0^{(1)}(x, \eta), x, 0\right), \\
&\Pi_0^{(1)}(x, 0) = -\overline{u}_0(x, 0), \qquad \Pi_0^{(1)}(x, \infty) = 0,
\end{aligned}$$

from which the solution is defined uniquely, and has the estimate [34, (2.2)]:

$$\left|\Pi_0^{(1)}(x, \eta)\right| \leq C e^{-\alpha \eta}, \tag{2.3}$$

where $\alpha > 0$ is a constant.

$\Pi_k^{(1)}$, $k \geq 1$, satisfies:

$$\frac{\partial^2 \Pi_k^{(1)}}{\partial \eta^2} = F\left(\overline{u}_0(x,0) + \Pi_0^{(1)}(x,\eta), x, 0\right) \Pi_k^{(1)} + \pi_k^{(1)},$$
$$\Pi_k^{(1)}(x,0) = -\overline{u}_k(x,0), \quad \Pi_k^{(1)}(x,\infty) = 0,$$

where $\pi_k^{(1)}$ depend on $\Pi_j^{(1)}$, where $j < k$. Also, $\Pi_k^{(1)}(x,\eta)$ have estimates of (2.3).

The other boundary correction functions $\Pi_k^{(2)}, \Pi_k^{(3)}, \Pi_k^{(4)}$, $k \geq 0$, are determined similarly. They all have estimates of the type (2.3).

The functions $P_k^{(i)}$ eliminate the discrepancies near the four corners of Ω, introduced by the Π-functions. $P_0^{(1)}$ satisfies the following relation [34, p. 1344]:

$$\frac{\partial^2 P_0^{(1)}}{\partial \xi^2} + \frac{\partial^2 P_0^{(1)}}{\partial \eta^2} = P_0^{(1)} F,$$
$$P_0^{(1)}(0,\eta) = -\omega(\eta), \qquad P_0^{(1)}(\xi,0) = -\omega(\xi),$$
$$P_0^{(1)}(\xi,\eta) \to 0 \text{ as } (\xi+\eta) \to 0.$$

Here $P_0^{(1)} F = F(\overline{u}_0 + \omega_1 + \omega_2 + P_0^{(1)}) - F(\overline{u}_0 + \omega_1) - F(\overline{u}_0 + \omega_2)$, where $F(u)$ is a shorthand notation of $F(u,0,0)$ and $\overline{u}_0 = \overline{u}_0(0,0), \omega_1 = \Pi_0^{(1)}(0,\eta) = \omega(\eta), \omega_2 = \Pi_0^{(2)}(\xi,0) = \omega(\xi)$, and $\omega(t)$ is a solution of the problem

$$\frac{d^2\omega}{dt^2} = F\left(\overline{u}_0 + \omega\right), \quad \omega(0) = -\overline{u}_0, \qquad \omega(\infty) = 0. \tag{2.4}$$

Also, $P_0^{(1)}(\xi,\eta)$ is bounded by

$$\left|P_0^{(1)}(\xi,\eta)\right| \leq C e^{-\alpha(\xi+\eta)}. \tag{2.5}$$

The function $P_k^{(1)}$, $k \geq 1$, satisfies the following:

$$\frac{\partial^2 P_k^{(1)}}{\partial \xi^2} + \frac{\partial^2 P_k^{(1)}}{\partial \eta^2} = F_u\left(\overline{u}_0 + \omega_1 + \omega_2 + P_0^{(1)}\right) P_k^{(1)} + h_k^{(1)},$$
$$P_k^{(1)}(0,\eta) = -\Pi_k^{(1)}(0,\eta), \qquad P_k^{(1)}(\xi,0) = -\Pi_k^{(2)}(\xi,0),$$
$$P_k^{(1)}(\xi,\eta) \to 0 \text{ as } (\xi+\eta) \to 0.$$

The functions $h_k^{(1)}$ here depend on $P_j^{(1)}$, where $j < k$. Also, the solution $P_k^{(1)}$ exists and has an estimate of the type (2.5).

The other corner correction functions $P_k^{(2)}, P_k^{(3)}, P_k^{(4)}$, $k \geq 0$, are determined in the same way. They all have estimates of the type (2.5).

From Boglayev [33], we have the following estimates.

LEMMA 2.2. *(See [33, Lemma 1].) For the solution $u(x,y) \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ of the problem (1.1),(1.2), we have*

$$\max_{(x,y)\in\overline{\Omega}} |u(x,y)| \leq m_1^{-1} \max_{(x,y)\in\overline{\Omega}} |F(u(x,y),x,y)|. \tag{2.6}$$

LEMMA 2.3. *(See [33, Lemma 2].) Let $u(x,y) \in C^2(\overline{\Omega}) \cap C^4(\Omega)$ be the solution of the problem (1.1),(1.2). Then the derivatives of u satisfy the following error estimates:*

$$\text{(I)} \qquad |u_{x^n}(x,y)| \leq C\left(1 + \varepsilon^{-n} e^{-\beta x/\varepsilon} + \varepsilon^{-n} e^{-\beta(1-x)/\varepsilon}\right), \qquad \text{on } \overline{\Omega},$$
$$\text{(II)} \qquad |u_{y^n}(x,y)| \leq C\left(1 + \varepsilon^{-n} e^{-\beta y/\varepsilon} + \varepsilon^{-n} e^{-\beta(1-y)/\varepsilon}\right), \qquad \text{on } \overline{\Omega},$$

where $0 < \beta < m_1^{1/2}$ and $n = 1, 2$.

Even though α is not clearly stated in [34], it is not difficult to see from Denisov's proof and Boglayev's proof [33] that α can be any constant such that $0 < \alpha < m_1^{1/2}$.

3. THE MESH AND SCHEME

Since this problem has boundary layers located along all sides of the rectangle of Ω, our piecewise uniform mesh can be constructed in the same way as we did for the linear problem [31]. Details can be found there.

Assume that the positive integers N_x and N_y are divisible by 4, where N_x and N_y denote the number of elements in the x- and y-directions, respectively. In the x-direction, we first divide the interval $[0,1]$ into the subintervals

$$[0,\sigma_x], \qquad [\sigma_x, 1-\sigma_x], \qquad [1-\sigma_x, 1].$$

Uniform meshes are then used on each subinterval, with $N_x/4$ points on each of $[0,\sigma_x]$ and $[1-\sigma_x, 1]$, and $N_x/2$ points on $[\sigma_x, 1-\sigma_x]$, where $\sigma_x = 2\alpha^{-1}\varepsilon \ln N_x$. Here, for simplicity, we assume that $\sigma_x \leq 1/4$, since we are considering SPP where ε is very small.

In the y-direction, we follow the same method described above by dividing the interval $[0,1]$ into the subintervals

$$[0,\sigma_y], \qquad [\sigma_y, 1-\sigma_y], \qquad [1-\sigma_y, 1].$$

Uniform meshes are then used on each subinterval, with $N_y/4$ points on each of $[0,\sigma_y]$ and $[1-\sigma_y, 1]$, and $N_y/2$ points on $[\sigma_y, 1-\sigma_y]$, where $\sigma_y = 2\alpha^{-1}\varepsilon \ln N_y$.

Let $I_i = [x_{i-1}, x_i]$, $I = [0,1]$, $\tilde{I}_i = I_i \times I$, $h = \max_{1\leq i\leq N_x} h_i$, $K_j = [y_{j-1}, y_j]$, $\tilde{K}_j = I \times K_j$, and $k = \max_{1\leq j\leq N_y} k_j$. Here $h_i = x_i - x_{i-1}$ and $k_j = y_j - y_{j-1}$.

The weak formulation of (1.1),(1.2) is: find $u \in H_0^1(\Omega)$ such that

$$\varepsilon^2 (\nabla u, \nabla v) + (F(u,x,y), v) = 0, \qquad \forall\, v \in H_0^1(\Omega), \tag{3.1}$$

where $(\cdot,\cdot)$ denote the usual $L^2(\Omega)$ inner product and $H_0^1(\Omega)$ is the usual Sobolev space.

Let $S_h(\Omega)$ be the ordinary bilinear finite element space [35]. Let

$$\Pi w \equiv \sum_{i=0}^{N_x} \sum_{j=0}^{N_y} w_{ij} l_i(x) l_j(y)$$

be the standard bilinear interpolate of w, where Π_x and Π_y are the interpolants in the x- and y-directions respectively. Here $l_i(x)$ is the so-called linear finite element function [35]. We seek the finite element solution $u_h \in S_h(\Omega)$ such that

$$\varepsilon^2 (\nabla u_h, \nabla v_h) + (F(u_h,x,y), v_h) = 0, \qquad \forall\, v_h \in S_h(\Omega). \tag{3.2}$$

Let us recall some results in [35], which will be used in this paper.

LEMMA 3.1. *(See [35, Theorem 2.1].)* $\Pi w = \Pi_x \Pi_y w = \Pi_y \Pi_x w$.

LEMMA 3.2. *(See [35, Theorem 2.6].)* $\|w - \Pi_x w\|_{\infty,\tilde{I}_i} \leq \frac{1}{8} h_i^2 \|w_{xx}\|_{\infty,\tilde{I}_i}$.

LEMMA 3.3. *(See [35, Lemma 2.1].)*

$$\|\Pi_x u\|_{\infty,\tilde{I}_i} \leq \max_{y\in I} \left(|u(x_{i-1},y)|, |u(x_i,y)|\right),$$
$$\|\Pi_x u\|_{\infty,\Omega} \leq \|u\|_{\infty,\Omega}.$$

The results obtained in Lemmas 2.3 and 3.2 hold true for the interpolant Π_y in the y-direction.

4. MAIN RESULTS

Using the techniques developed in [29,31], we can obtain the following interpolation estimates.

LEMMA 4.1. *For a sufficiently smooth solution u of (1.1),(1.2) and any integer $n \geq 0$, we have*

$$\text{(I)} \qquad \|u - \Pi_x u\|_{\infty,\tilde{I}_i} \leq C\left(N_x^{-2}\ln^2 N_x + \varepsilon^{n+1} + \sum_{k=0}^{n} \varepsilon^k N_x^{-2}\right), \qquad \forall i = 1,\dots,N_x,$$

$$\text{(II)} \qquad \|u - \Pi_y u\|_{\infty,\tilde{K}_j} \leq C\left(N_y^{-2}\ln^2 N_y + \varepsilon^{n+1} + \sum_{k=0}^{n} \varepsilon^k N_y^{-2}\right), \qquad \forall j = 1,\dots,N_y.$$

PROOF. For $i = 1,\dots,i_0, N_x - i_0 + 1,\dots,N_x$, by Lemmas 2.3 and 3.2, we have

$$\begin{aligned}\|u - \Pi_x u\|_{\infty,\tilde{I}_i} &\leq Ch_i^2 \|u_{xx}\|_{\infty,\tilde{I}_i} \leq Ch_i^2 \max_{x\in I_i}\left(1 + \varepsilon^{-2}e^{-\beta x/\varepsilon} + \varepsilon^{-2}e^{-\beta(1-x)/\varepsilon}\right)\\ &\leq Ch_i^2\left(1+\varepsilon^{-2}\right) \leq CN_x^{-2}\ln^2 N_x,\end{aligned}$$

since $h_i = 4\sigma_x/N_x$ in this case. Hence, (I) is true in this case.

For $i = i_0+1,\dots,N_x - i_0$, in which case $[x_{i-1}, x_i] \subseteq [\sigma_x, 1-\sigma_x]$. We can write $\Pi_x u$ as

$$\Pi_x u = \Pi_x U_n + \Pi_x\left(u - U_n\right). \tag{4.1}$$

By Lemmas 3.3 and 2.1, we have

$$\|\left(u - U_n\right) - \Pi_x\left(u - U_n\right)\|_{\infty,\tilde{I}_i} \leq 2\|u - U_n\|_{\infty,\tilde{I}_i} \leq C\varepsilon^{n+1} \tag{4.2}$$

In the following, we will estimate $\Pi_x U_n$ by using the asymptotic expansion (2.1). Note that $\overline{u}_k(x,y)$ are independent of ε, by Lemma 3.2, we have

$$\|\overline{u}_k(x,y) - \Pi_x\overline{u}_k(x,y)\|_{\infty,\tilde{I}_i} \leq Ch_i^2\,\|(\overline{u}_k(x,y))_{xx}\|_{\infty,\tilde{I}_i} \leq CN_x^{-2}, \tag{4.3}$$

where in the last step we used the fact that

$$N_x^{-1} \leq h_i \leq 2N_x^{-1}, \qquad \text{for } i = i_0+1,\dots,N_x - i_0.$$

By the same arguments, we obtain

$$\left\|\Pi_k^{(1)}(x,\eta) - \Pi_x\Pi_k^{(1)}(x,\eta)\right\|_{\infty,\tilde{I}_i} \leq CN_x^{-2}, \tag{4.4}$$

$$\left\|\Pi_k^{(3)}(x,\eta_*) - \Pi_x\Pi_k^{(3)}(x,\eta_*)\right\|_{\infty,\tilde{I}_i} \leq CN_x^{-2}. \tag{4.5}$$

By Lemma 3.3, we have

$$\left\|\Pi_k^{(2)}(\xi,y) - \Pi_x\Pi_k^{(2)}(\xi,y)\right\|_{\infty,\tilde{I}_i} \leq 2\|\Pi_k^{(2)}(\xi,y)\|_{\infty,\tilde{I}_i} \tag{4.6}$$

$$\leq Ce^{-\alpha\xi_{i-1}} \leq Ce^{-\alpha\sigma_x/\varepsilon} = CN_x^{-2}. \tag{4.7}$$

Similarly, we have

$$\left\|\Pi_k^{(4)}(\xi_*,y) - \Pi_x\Pi_k^{(4)}(\xi_*,y)\right\|_{\infty,\tilde{I}_i} \leq CN_x^{-2}. \tag{4.8}$$

By Lemma 3.3, we have

$$\left\|P_k^{(1)}(\xi,\eta) - \Pi_x P_k^{(1)}(\xi,\eta)\right\|_{\infty,\tilde{I}_i} \leq 2\left\|P_k^{(1)}(\xi,\eta)\right\|_{\infty,\tilde{I}_i} \tag{4.9}$$

$$\leq Ce^{-\alpha\xi_{i-1}} \leq Ce^{-\alpha\sigma_x/\varepsilon} = CN_x^{-2}. \tag{4.10}$$

Using the same reasoning, we can obtain

$$\left\|P_k^{(2)}(\xi,\eta_*) - \Pi_x P_k^{(2)}(\xi,\eta_*)\right\|_{\infty,\tilde{I}_i} \le C N_x^{-2}, \tag{4.11}$$

$$\left\|P_k^{(3)}(\xi_*,\eta_*) - \Pi_x P_k^{(3)}(\xi_*,\eta_*)\right\|_{\infty,\tilde{I}_i} \le C N_x^{-2}, \tag{4.12}$$

$$\left\|P_k^{(4)}(\xi_*,\eta) - \Pi_x P_k^{(4)}(\xi_*,\eta)\right\|_{\infty,\tilde{I}_i} \le C N_x^{-2}. \tag{4.13}$$

By combining the above inequalities, we have

$$\|U_n - \Pi_x U_n\|_{\infty,\tilde{I}_i} \le C \sum_{k=0}^{n} \varepsilon^k N_x^{-2}, \tag{4.14}$$

which together with (4.2) concludes our proof.

The proof of (II) can be carried out in the same way as (I). ∎

Therefore, we have the following.

LEMMA 4.2. *For the solution u of (1.1),(1.2) and any integer $n \ge 0$, we have*

$$\|u - \Pi u\|_{\infty,\Omega} \le C \left(N_x^{-2} \ln^2 N_x + N_y^{-2} \ln^2 N_y + \varepsilon^{n+1} + \sum_{k=0}^{n} \varepsilon^k \left(N_x^{-2} + N_y^{-2}\right) \right).$$

PROOF. By Lemmas 3.1, 3.2, and 4.1, we have the following.

$$\begin{aligned} \|u - \Pi u\|_{\infty,\Omega} &\le \|u - \Pi_x u\|_{\infty,\Omega} + \|\Pi_x (u - \Pi_y u)\|_{\infty,\Omega} \\ &\le \|u - \Pi_x u\|_{\infty,\Omega} + \|u - \Pi_y u\|_{\infty,\Omega} && (4.15) \\ &\le \max_{1\le i\le N_x} \|u - \Pi_x u\|_{\infty,\tilde{I}_i} + \max_{1\le j\le N_y} \|u - \Pi_y u\|_{\infty,\tilde{K}_j} && (4.16) \\ &\le C\left(N_x^{-2} \ln^2 N_x + N_y^{-2} \ln^2 N_y + \varepsilon^{n+1} + \sum_{k=0}^{n} \varepsilon^k \left(N_x^{-2} + N_y^{-2}\right)\right), && (4.17) \end{aligned}$$

which concludes our proof. ∎

THEOREM 4.3. *Let u_h be the finite element solution of (3.2), and u be the analytic solution of (1.1),(1.2). Then we have*

$$\|u - u_h\| \le C \left(1 + \varepsilon N_x + \varepsilon^{1/2} N_x \ln^{-1/2} N_x + \varepsilon N_y + \varepsilon^{1/2} N_y \ln^{-1/2} N_y\right) \|u - \Pi u\|_{\infty,\Omega}. \tag{4.18}$$

PROOF. By subtracting (3.2) from (3.1), we have

$$\varepsilon^2 \left(\nabla (u - u_h), \nabla v_h\right) + \left(F(u,x,y) - F(u_h,x,y), v_h\right) = 0, \qquad \forall v_h \in S_h(\Omega). \tag{4.19}$$

By the mean value theorem, we can rewrite (4.19) as

$$\varepsilon^2 \left(\nabla (u - u_h), \nabla v_h\right) + \left(\tilde{F}_u \cdot (u - u_h), v_h\right) = 0, \qquad \forall v_h \in S_h(\Omega), \tag{4.20}$$

where $\tilde{F}_u$ denotes the value of the derivative F_u at some point $\theta u + (1-\theta)u_h$, $0 < \theta < 1$.

From (4.20), we have

$$\begin{aligned} &\varepsilon^2 \left(\nabla (\Pi u - u_h), \nabla v_h\right) + \left(\tilde{F}_u \cdot (\Pi u - u_h), v_h\right) && (4.21) \\ &= \varepsilon^2 \left(\nabla (\Pi u - u), \nabla v_h\right) + \left(\tilde{F}_u \cdot (\Pi u - u), v_h\right), \qquad \forall v_h \in S_h(\Omega). && (4.22) \end{aligned}$$

By denoting $\chi = \Pi u - u_h$, choosing $v_h = \chi$ in (4.21), and using Assumption (A2), we can obtain

$$\varepsilon^2 \|\nabla\chi\|^2 + m_1 \|\chi\|^2 \le \varepsilon^2 |(\nabla(\Pi u - u), \nabla v_h)| + \left|\left(\tilde{F}_u \cdot (\Pi u - u), v_h\right)\right|. \tag{4.23}$$

Integrating by parts, we obtain

$$\begin{aligned}
\varepsilon^2 ((\Pi u - u)_x, \chi_x) &= \sum_{i,j} \int_{x_{i-1}}^{x_i} \int_{y_{j-1}}^{y_j} \varepsilon^2 (\Pi u - u)_x \chi_x \, dx\, dy \\
&= \sum_{i,j} \int_{y_{j-1}}^{y_j} \varepsilon^2 (\Pi u - u)|_{x=x_{i-1}}^{x=x_i} \chi_x \, dy \\
&\le \sum_{i,j} 2 \int_{y_{j-1}}^{y_j} |\varepsilon \chi_x| \, dy \cdot \varepsilon \|\Pi u - u\|_{\infty,\Omega} \\
&= \sum_{i,j} \frac{2}{x_i - x_{i-1}} \int_{X_{i-1}}^{x_i} \int_{y_{j-1}}^{y_j} |\varepsilon \chi_x| \, dy\, dx \cdot \varepsilon \|\Pi u - u\|_{\infty,\Omega},
\end{aligned}$$

where $\sum_{i,j}$ is a short notation for $\sum_{1\le i\le N_x, 1\le j\le N_y}$.

Note that

$$\begin{aligned}
&\sum_{i,j} \frac{1}{x_i - x_{i-1}} \int_{x_{i-1}}^{x_i} \int_{y_{j-1}}^{y_j} |\varepsilon \chi_x| \, dy\, dx \\
&\le N_x \int_{S_1} |\varepsilon \chi_x| \, dy\, dx + \frac{N_x}{\varepsilon \ln N_x} \int_{S_2} |\varepsilon \chi_x| \, dy\, dx \\
&\le N_x (\text{meas}(S_1))^{1/2} \|va\chi_x\|_{L^2(S_1)} + \frac{N_x}{\varepsilon \ln N_x} (\text{meas}(S_2))^{1/2} \|\varepsilon\chi_x\|_{L^2(S_2)} \\
&\le N_x \|\varepsilon\chi_x\|_{L^2(S_1)} + C \frac{N_x}{\varepsilon \ln N_x} (\varepsilon \ln N_x)^{1/2} \|\varepsilon\chi_x\|_{L^2(S_2)} \\
&\le C \left(N_x + \varepsilon^{-1/2} N_x \ln^{-1/2} N_x\right) \|\varepsilon\chi_x\|,
\end{aligned}$$

where $S_1 \equiv [\sigma_x, 1 - \sigma_x] \times [0, 1]$ and $S_2 \equiv \overline{\Omega} \backslash S_1$.

Therefore we have,

$$\varepsilon^2 ((\Pi u - u)_x, \chi_x) \le C \left(\varepsilon N_x + \varepsilon^{1/2} N_x \ln^{-1/2} N_x\right) \|\Pi u - u\|_{\infty,\overline{\Omega}} \|\varepsilon\chi_x\|. \tag{4.24}$$

Similarly, we can obtain

$$\varepsilon^2 \left((\Pi u - u)_y, \chi_y\right) \le C \left(\varepsilon N_y + \varepsilon^{1/2} N_y \ln^{-1/2} N_y\right) \|\Pi u - u\|_{\infty,\overline{\Omega}} \|\varepsilon\chi_y\|. \tag{4.25}$$

By Assumption (A2), we have

$$\left(\tilde{F}_u \cdot (\Pi u - u), \chi\right) \le m_2 \|\Pi u - u\|_{\infty,\Omega} \cdot \|\chi\|. \tag{4.26}$$

Combining the above inequalities, we have

$$\varepsilon^2 \|\nabla(\Pi u - u_h)\| + \|\Pi u - u_h\| \tag{4.27}$$

$$\le C \left(1 + \varepsilon N_x + \varepsilon^{1/2} N_x \ln^{-1/2} N_x + \varepsilon N_y + \varepsilon^{1/2} N_y \ln^{-1/2} N_y\right) \|\Pi u - u\|_{\infty,\overline{\Omega}}, \tag{4.28}$$

which along with the triangular inequality completes our proof. ∎

Since we consider here singularly perturbed problems, the parameter ε is usually very small. Under the assumption

(A3) $\varepsilon \le \max(N_x^{-2} \ln N_x, N_y^{-2} \ln N_y)$,

and letting $n = 0$ in the asymptotic expansion (2.1), we can easily obtain the following quasi-optimal global uniformly convergent result.

COROLLARY 4.4. *Let u_h be the finite element solution of (3.2) and u be the solution of (1.1),(1.2). Then under Assumptions (A1), (A2), and (A3), we have*

$$\|u - u_h\| \le C\left(N_x^{-2}\ln^2 N_x + N_y^{-2}\ln^2 N_y\right),$$

where the constant C is independent of the perturbation parameter ε.

5. FURTHER DISCUSSION

To solve the nonlinear equation (3.2) efficiently, we consider hereby an iterative scheme [36, p. 59]:

$$\varepsilon^2\left(\nabla u_h^{m+1}, \nabla v_h\right) + \left(F\left(u_h^m, x, y\right), v_h\right) + \lambda\left(u_h^{m+1} - u_h^m, v_h\right) = 0, \qquad \forall\, v_h \in S_h(\Omega),$$

where $\lambda > 0$ is a parameter to be chosen later.

Let $z^{m+1} = u_h^{m+1} - u_h^m$. By the mean value theorem, we have

$$\varepsilon^2\left(\nabla z^{m+1}, \nabla v_h\right) + \left(\tilde{F}_u^m \cdot z^m, v_h\right) + \lambda\left(z^{m+1} - z^m, v_h\right) = 0, \qquad \forall\, v_h \in S_h(\Omega), \tag{5.1}$$

where $\tilde{F}_u^m$ denotes the value of F_u at some point $\theta_1 u_h^m + (1-\theta_1)u_h^{m-1}$, $0 < \theta_1 < 1$.

Let $v_h = z^{m+1}$ in (5.1), we have

$$\varepsilon^2\left\|\nabla z^{m+1}\right\|^2 + \lambda\left\|z^{m+1}\right\|^2 + \left(\left(\tilde{F}_u^m - \lambda\right) z^m, z^{m+1}\right) = 0, \tag{5.2}$$

from which we have

$$\varepsilon^2\left\|\nabla z^{m+1}\right\|^2 + \lambda\left\|z^{m+1}\right\|^2 = \left(\left(\lambda - \tilde{F}_u^m\right) z^m, z^{m+1}\right) \tag{5.3}$$

$$\le \sup\left|\lambda - \tilde{F}_u^n\right| \cdot \left\|z^{m+1}\right\| \cdot \left\|z^m\right\|. \tag{5.4}$$

By Assumption (A2), if we choose $\lambda = 2m_2$, then we have

$$\left\|z^{m+1}\right\| \le \frac{1}{2}\left\|z^m\right\|, \tag{5.5}$$

from which we see that the functions $u_h^m, m = 0, 1, \dots,$ form a Cauchy sequence and converge to the finite element solution of (3.2). The uniqueness of the finite element solution of (3.2) can be proved in the same way as [36, p. 61].

A numerical experiment for the linear case (which is a special case of (1.1),(1.2)) was carried out in [31], which is consistent with our theoretical convergence rate (4.29).

As Roos [27, Section 2.1.3] mentioned, "not much is known about Shishkin-type grids for nonlinear problems". This paper is the first to generalize our linear techniques [29,31] to nonlinear partial differential equations for such Shishkin-type grids. It is not difficult to see that our methods can be generalized to more complicated nonlinear problems only if they have similar asymptotic expansions as (2.1).

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