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# Uniformly Convergent Finite Element Methods for Singularly Perturbed Elliptic Boundary Value Problems I: Reaction-Diffusion Type

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**Abstract**—We consider the bilinear finite element method on a Shishkin mesh for the singularly perturbed elliptic boundary value problem $-\varepsilon^2(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}) + a(x,y)u = f(x,y)$ in two space dimensions. By using a very sophisticated asymptotic expansion of Han *et al.* [1] and the technique we used in [2], we prove that our method achieves almost second-order uniform convergence rate in $L^2$-norm. Numerical results confirm our theoretical analysis.

**Keywords**—Finite element methods, Singularly perturbed problems, Elliptic partial differential equations.

## 1. INTRODUCTION

In this paper, we will consider a finite element method (FEM) for the singularly perturbed elliptic boundary value problem:

$$L_\varepsilon u \equiv -\varepsilon^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + a(x,y)u = f(x,y), \qquad \text{in } \Omega \equiv (0,1) \times (0,1), \tag{1}$$

$$u = 0, \qquad \text{on } \partial\Omega, \tag{2}$$

where $\varepsilon \in (0,1]$ is a small positive parameter. The functions $a$ and $f$ are assumed to be sufficiently smooth in $\Omega$ and

$$a(x,y) \geq \alpha^2 > 0, \qquad \text{in } \Omega.$$

Singularly perturbed problems appear in many branches of applied mathematics. While the problem can be traced back to 1904 when Prandtl introduced the terminology *boundary layer* at the Third International Congress of Mathematicians in Heidelberg, it still constitutes a very active research area, as evidenced by recent books, such as [3–5]. It is well known that singularly perturbed problems often have very thin boundary layers and internal layers, cf. [3–5]. These

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Typeset by $\mathcal{A}_\mathcal{M}\mathcal{S}$-TEX problems are very difficult to solve numerically, cf. [5–7]. While a suitable amount of work has been carried out, a large number of unsolved problems still remain to be solved, especially for two-dimensional problems, cf. [5].

In the context of the finite element method for singularly perturbed problems, a large number of different methods have been investigated. For instance, *Streamline Diffusion FEM* [8,9], *High-order FEM* [10,11], *Adaptive FEM* [12], and *Discontinuous Galerkin method* [8], to name but a few. For more details see [5;13, Chapter 12]. But few of the above methods are uniformly convergent; that is, the error between the original solution and the computed FEM solution $u_h$ satisfies:

$$||u - u_h||_\varepsilon \le ch^m$$

for some positive constant $m$ that is independent of $\varepsilon$ and of the mesh.

As far as we know, uniform convergence can be achieved by using exponentially fitted splines or combinations with other functions as trial and test space, cf. [14,15]. However they have very low convergence rate, which is $||u - u_h||_\varepsilon \le ch^{1/2}$, where $||\cdot||_\varepsilon$ is a variant of an energy norm. Another type of uniform convergence is achieved by using hp FEM [11]. In [11], Schwab *et al.* considered only the simplest case $f \equiv 1$ and tested it on several types of meshes. Surprisingly, their computational results show that the boundary layer (BL) mesh (other than BL-corner mesh and BL-refined corner mesh) performs best for the energy norm [11, p. 25]. There is no explanation for this phenomenon. Actually, the BL mesh is similar to the Shishkin mesh [7,16,17] in some sense. Recently, Guo, Sun, and Stynes [18,19] achieved almost optimal convergence results for ordinary differential equations and parabolic equations by using piecewise polynomial Galerkin FEM on a Shishkin mesh. These studies were carried out in one space dimension. Madden *et al.* [20] and Roos [21] carried out some computational experiments for FEM on Shishkin meshes in two dimensions, and the results seemed very promising. To our knowledge, there is no theoretical analysis for FEM on Shishkin meshes in two space dimensions. Just as Roos *et al.* said in [5]: "Finite element methods that use Shishkin meshes in two or more dimensions have not been explored in the literature."

This paper is our first attempt in the above mentioned area. Problem (1) is of the so-called reaction-diffusion type problem according to the classification of [5]. The investigation for a convection-diffusion type problem is under development [22]. By using a very sophisticated asymptotic expansion of Han *et al.* [1] and a technique we used in [2], we prove that the bilinear FEM on a Shishkin mesh achieves almost second-order uniform convergence rate in $L^2$-norm. Numerical results confirm our theoretical analysis.

The organization of this paper is as follows. In Section 2, we present the Butuzov asymptotic expansion [1,23,24] for the solution u of (1),(2). In Section 3, we develop the derivative estimates for $u$. Then in Section 4, we provide our FEM based on a Shishkin mesh. In Section 5, we prove that our FEM achieves the uniform convergence rate in both cases. Finally numerical results are provided and discussed in Section 6.

Through this paper we shall use $C$, sometimes subscripted, to denote a generic positive constant that is independent of both $\varepsilon$ and the mesh.

## 2. THE ASYMPTOTIC EXPANSION

In this section, we describe the Butuzov asymptotic expansion [1,23,24]. This section is based mostly on Han *et al.* [1]. Here we use $D^m u(x, y)$ to denote the generic derivative of order $m$ of $u$.

We start with the "outer expansion". Let $u_0(x, y) = f(x, y)/a(x, y)$, $u_1(x, y) = 0$ and $u_i(x, y) = \Delta u_{i-2}(x, y)/a(x, y)$ for $i \ge 2$. Hence, $u_i = 0$ for $i$ odd and

$$L_\varepsilon u_i = -\varepsilon^2 \Delta u_i + \Delta u_{i-2}, \qquad i = 2, 4, \dots,$$

where $\triangle$ is the Laplacian operator. Let

$$U_{2n}(x,y)=\sum_{i=0}^{2n}\varepsilon^i u_i(x,y).$$

Since $u-U_{2n}$ is not small on $\partial\Omega$, we have to introduce the boundary layer functions to correct the discrepancy between the boundary data and the boundary values of the reduced problem near the four boundary sides. Han *et al.* [1, pp. 396–397] constructed the following boundary layer functions:

$$\begin{aligned}
V_{2n}(x,\eta)&=\sum_{i=0}^{2n}\varepsilon^i v_i(x,\eta), &&\text{at side } y=0, &&\text{where } \eta=\frac{y}{\varepsilon},\\
W_{2n}(\xi,y)&=\sum_{i=0}^{2n}\varepsilon^i w_i(\xi,y), &&\text{at side } x=0, &&\text{where } \xi=\frac{x}{\varepsilon},\\
\overline{V}_{2n}(x,\overline{\eta})&=\sum_{i=0}^{2n}\varepsilon^i v_i(x,\overline{\eta}), &&\text{at side } y=1, &&\text{where } \overline{\eta}=\frac{(1-y)}{\varepsilon},\\
\overline{W}_{2n}\left(\overline{\xi},y\right)&=\sum_{i=0}^{2n}\varepsilon^i w_i\left(\overline{\xi},y\right), &&\text{at side } x=1, &&\text{where } \overline{\xi}=\frac{(1-x)}{\varepsilon}.
\end{aligned}$$

Since the remainder, $u-U_{2n}-V_{2n}-W_{2n}-\overline{V}_{2n}-\overline{W}_{2n}$, is not small near the four vertices of $\Omega$, Han *et al.* [1, pp. 397–398] introduced the following corner layer functions to correct this discrepancy in the boundary data near the four corners:

$$\begin{aligned}
Z_{2n}^1(\xi,\eta)&=\sum_{i=0}^{2n}\varepsilon^i z_i^1(\xi,\eta), &&\text{at corner } (0,0), &&\text{where } \xi=\frac{x}{\varepsilon},\quad \eta=\frac{y}{\varepsilon},\\
Z_{2n}^2\left(\overline{\xi},\eta\right)&=\sum_{i=0}^{2n}\varepsilon^i z_i^2\left(\overline{\xi},\eta\right), &&\text{at corner } (1,0), &&\text{where } \overline{\xi}=\frac{(1-x)}{\varepsilon},\quad \eta=\frac{y}{\varepsilon},\\
Z_{2n}^3\left(\xi,\overline{\eta}\right)&=\sum_{i=0}^{2n}\varepsilon^i z_i^3\left(\xi,\overline{\eta}\right), &&\text{at corner } (0,1), &&\text{where } \xi=\frac{x}{\varepsilon},\quad \overline{\eta}=\frac{(1-y)}{\varepsilon},\\
Z_{2n}^4\left(\overline{\xi},\overline{\eta}\right)&=\sum_{i=0}^{2n}\varepsilon^i z_i^4\left(\overline{\xi},\overline{\eta}\right), &&\text{at corner } (1,1), &&\text{where } \overline{\xi}=\frac{(1-x)}{\varepsilon},\quad \overline{\eta}=\frac{(1-y)}{\varepsilon}.
\end{aligned}$$

Also they showed the following results.

LEMMA 2.1. *(See [1, (2.6c) (2.9c)].) For the boundary layer functions defined above, we have*

$$\begin{aligned}
\left|D_{x\eta}^m V_{2n}(x,\eta)\right|&\le C_m e^{-\alpha\eta}, && m=0,1,\dots,\\
\left|D_{\xi y}^m W_{2n}(\xi,y)\right|&\le C_m e^{-\alpha\xi}, && m=0,1,\dots,\\
\left|D_{x\overline{\eta}}^m \overline{V}_{2n}(x,\overline{\eta})\right|&\le C_m e^{-\alpha\overline{\eta}}, && m=0,1,\dots,\\
\left|D_{\overline{\xi}y}^m \overline{W}_{2n}\left(\overline{\xi},y\right)\right|&\le C_m e^{-\alpha\overline{\xi}}, && m=0,1,\dots,\\
\left|Z_{2m}^1\left(\xi,\eta\right)\right|&\le Ce^{-\alpha(\xi+\eta)}, && m=0,1,\dots,\\
\left|Z_{2m}^2\left(\overline{\xi},\eta\right)\right|&\le Ce^{-\alpha(\overline{\xi}+\eta)}, && m=0,1,\dots,\\
\left|Z_{2m}^3\left(\xi,\overline{\eta}\right)\right|&\le Ce^{-\alpha(\xi+\overline{\eta})}, && m=0,1,\dots,\\
\left|Z_{2m}^4\left(\overline{\xi},\overline{\eta}\right)\right|&\le Ce^{-\alpha(\overline{\xi}+\overline{\eta})}, && m=0,1,\dots.
\end{aligned}$$

THEOREM 2.1. *(See [1, Theorem 21].) Let $u$ solve (1),(2). There is a constant $C_n > 0$ that is independent of $\varepsilon$ such that*

$$|R_{2n}(x,y)| \leq C_n \varepsilon^{2n+1},$$

*where $R_{2n} = u - \tilde{u}_{2n}$ denote the remainder in the asymptotic expansion*

$$\tilde{u}_{2n} = U_{2n} + V_{2n} + W_{2n} + \overline{V}_{2n} + \overline{W}_{2n} + \sum_{l=1}^{4} Z_{2n}^{l}.$$

## 3. DERIVATIVE ESTIMATES OF THE SOLUTION

In this section, we will obtain some derivative estimates for the solution of (1),(2). We assume the following compatibility conditions [5,15]:

$$f(0,0) = f(0,1) = f(1,0) = f(1,1) = 0$$

which ensure that the solution of (1),(2) $u(x,y) \in C^4(\Omega) \cap C^2(\overline{\Omega})$, where $\overline{\Omega} = \Omega \cup \partial\Omega$. Such compatibility conditions are necessary for the pointwise derivative estimates of the solution [2,15,25]. Here we will make repeated use of the following weak maximum principle.

THEOREM 3.1. *For any functions $w(x,y) \in C^2(\Omega) \cap C^0(\overline{\Omega})$, if $w \geq 0$ on $\partial\Omega$ and $L_\varepsilon w \geq 0$ on $\Omega$, then $w \geq 0$ on $\overline{\Omega}$.*

PROOF. It can be proved easily by contradiction, cf. Gilberg *et al.* [26, Theorem 3.1].

By choosing the barrier functions [2,15,25] properly, we can obtain the following estimates for the solution $u$ of (1),(2).

LEMMA 3.1.

(I) $|u(x,y)| \leq C(1 - e^{-\alpha x/\varepsilon})$, *on* $\overline{\Omega}$,
(II) $|u(x,y)| \leq C(1 - e^{-\alpha(1-x)/\varepsilon})$, *on* $\overline{\Omega}$,
(III) $|u(x,y)| \leq C(1 - e^{-\alpha y/\varepsilon})$, *on* $\overline{\Omega}$,
(IV) $|u(x,y)| \leq C(1 - e^{-\alpha(1-y)/\varepsilon})$, *on* $\overline{\Omega}$.

PROOF.

(I) Use the barrier function $\phi(x,y) = C(1 - e^{-\alpha x/\varepsilon})$, then we have

$$\begin{aligned} L_\varepsilon(\phi \pm u) &= C\alpha^2 e^{-\alpha x/\varepsilon} + aC\left(1 - e^{-\alpha x/\varepsilon}\right) \pm f, \\ &= C\left(\alpha^2 - a\right)\left(e^{-\alpha x/\varepsilon} - 1\right) + C\alpha^2 \pm f. \end{aligned}$$

Note that

$$\left(\alpha^2 - a\right)\left(e^{-\alpha x/\varepsilon} - 1\right) \geq 0.$$

Hence,

$$L_\varepsilon(\phi \pm u) \geq C\alpha^2 \pm f \geq 0, \qquad \text{for } C \text{ sufficiently large,}$$

then from $(\phi \pm u)|_{\partial\Omega} \geq 0$ and Theorem 3.1 concludes our proof.

(II) Use the barrier function $\phi(x,y) = C(1 - e^{-\alpha(1-x)/\varepsilon})$.
(III) Use the barrier function $\phi(x,y) = C(1 - e^{-\alpha y/\varepsilon})$.
(IV) Use the barrier function $\phi(x,y) = C(1 - e^{-\alpha(1-y)/\varepsilon})$.

LEMMA 3.2.

(I) $|u_x(x,y)| \le C\varepsilon^{-1}$, *on* $\partial\Omega$,

(II) $|u_y(x,y)| \le C\varepsilon^{-1}$, *on* $\partial\Omega$.

PROOF.

(I) By Lemma 3.1 , we have

$$|u_x(0,y)| = \left|\lim_{x\to 0^+} \frac{u(x,y)-u(0,y)}{x}\right| \le \lim_{x\to 0^+}\left|\frac{u(x,y)-u(0,y)}{x}\right|$$
$$\le \lim_{x\to 0^+} \frac{C\left(1-e^{-\alpha x/\varepsilon}\right)}{x} = C\frac{\alpha}{\varepsilon} \le C\varepsilon^{-1}.$$

By Lemma 3.1(II), we can get the estimate for $u_x(1,y)$ in a similar way.

Differentiating the given boundary conditions $u(x,y) = 0$ at $y = 0$ and $y = 1$ with respect to $x$ gives us $u_x(x,0) = u_x(x,1) = 0$, from which finishes our proof.

(II) Use the similar proof as (I) by Lemma 3.1(III) and (IV).

LEMMA 3.3.

(I) $|u_x(x,y)| \le C\left(1+\varepsilon^{-1}e^{-\alpha x/\varepsilon}+\varepsilon^{-1}e^{-\alpha(1-x)/\varepsilon}\right)$, *on* $\overline{\Omega}$,

(II) $|u_y(x,y)| \le C\left(1+\varepsilon^{-1}e^{-\alpha y/\varepsilon}+\varepsilon^{-1}e^{-\alpha(1-y)/\varepsilon}\right)$, *on* $\overline{\Omega}$.

PROOF.

(I) Consider the barrier function $\phi(x,y) = C(1+\varepsilon^{-1}e^{-\alpha x/\varepsilon}+\varepsilon^{-1}e^{-\alpha(1-x)/\varepsilon})$, then we have

$$\begin{aligned} L_\varepsilon(\phi \pm u_x) \ge & -\alpha^2 C\left(\varepsilon^{-1}e^{-\alpha x/\varepsilon}+\varepsilon^{-1}e^{-\alpha(1-x)/\varepsilon}\right) \\ & + aC\left(1+\varepsilon^{-1}e^{-\alpha x/\varepsilon}+\varepsilon^{-1}e^{-\alpha(1-x)/\varepsilon}\right) \pm (f_x - a_x u) \\ \ge & \; aC \pm (f_x - a_x u) \ge 0, \qquad \text{for } C \text{ sufficiently large,} \end{aligned}$$

and note that $(\phi \pm u_x)|_{\partial\Omega} \ge 0$, from which concludes our proof of (1).

(II) To prove (II), use the barrier function $\phi(x,y) = C(1+\varepsilon^{-1}e^{-\alpha y/\varepsilon}+\varepsilon^{-1}e^{-\alpha(1-y)/\varepsilon})$.

LEMMA 3.4.

(I) $|u_{xx}(x,y)| \le C\varepsilon^{-2}$, on $\partial\Omega$,

(II) $|u_{yy}(x,y)| \le C\varepsilon^{-2}$, on $\partial\Omega$.

PROOF.

(I) Due to the boundary conditions $u(x,y) = 0$ at $y = 0$ and $y = 1$, we have $u_{xx} = 0$ at $y = 0$ and $y = 1$. By setting $x = 0, 1$ in (1) and using the fact that $u = u_{yy} = 0$ on the sides of $x = 0$ and $x = 1$, we have $u_{xx} = 0$ at $x = 0$ and $x = 1$.

(II) Use a similar proof as in (I).

LEMMA 3.5.

(I) $|u_{xx}(x,y)| \le C\left(1+\varepsilon^{-2}e^{-\alpha x/\varepsilon}+\varepsilon^{-2}e^{-\alpha(1-x)/\varepsilon}\right)$, on $\overline{\Omega}$,

(II) $|u_{yy}(x,y)| \le C\left(1+\varepsilon^{-2}e^{-\alpha x/\varepsilon}+\varepsilon^{-2}e^{-\alpha(1-x)/\varepsilon}\right)$, on $\overline{\Omega}$.

PROOF.

(I) Use the barrier function $\phi(x,y) = C(1+\varepsilon^{-2}e^{-\alpha x/\varepsilon}+\varepsilon^{-2}e^{-\alpha(1-x)/\varepsilon})$, then

$$\begin{aligned} L_\varepsilon\left(\phi \pm u_{xx}\right) \ge & -\alpha^2 C\left(\varepsilon^{-2}e^{-\alpha x/\varepsilon}+\varepsilon^{-2}e^{-\alpha(1-x)/\varepsilon}\right) \\ & + aC\left(1+\varepsilon^{-2}e^{-\alpha x/\varepsilon}+\varepsilon^{-2}e^{-\alpha(1-x)/\varepsilon}\right) \pm (f_{xx} - a_{xx}u - 2a_x u_x) \\ \ge & \; aC \pm (f_{xx} - a_{xx}u - 2a_x u_x) \ge 0, \qquad \text{for } C \text{ sufficiently large,} \end{aligned}$$

then from $(\phi \pm u_{xx})|_{\partial\Omega} \ge 0$ and Theorem 3.1 concludes our proof.

(II) Use the barrier function $\phi(x,y) = C(1+\varepsilon^{-2}e^{-\alpha x/\varepsilon}+\varepsilon^{-2}e^{-\alpha(1-x)/\varepsilon})$.

# 4. FINITE ELEMENT METHOD ON SHISHKIN MESH

To construct a Shishkin mesh, we assume that the positive integers $N_x$ and $N_y$ are divisible by 4, where $N_x, N_y$ denote the number of mesh points in the $x$- and $y$-directions, respectively. In $x$-direction, we can construct the Shishkin mesh by dividing the interval $[0, 1]$ into the subintervals

$$[0, \sigma_x], \qquad [\sigma_x, 1-\sigma_x], \qquad [1-\sigma_x, 1].$$

Uniform meshes are then used on each subinterval, with $N_x/4$ points on each of $[0, \sigma_x]$ and $[1-\sigma_x, 1]$, and $N_x/2$ points on $[\sigma_x, 1-\sigma_x]$. Here, $\sigma_x$ is defined by $\sigma_x = \min\{1/4, 2\alpha^{-1}\varepsilon \ln N_x\}$. More explicitly, we have

$$0 = x_0 < x_1 < \cdots < x_{i_0} < \cdots < x_{N_x - i_0} < \cdots < x_{N_x} = 1,$$

with $i_0 = N_x/4$, $x_{i_0} = \sigma_x$, $x_{N_x-i_0} = 1 - \sigma_x$, and

$$\begin{aligned} h_i &= 4\sigma_x N_x^{-1}, && \text{for } i = 1, \cdots, i_0, \quad N_x - i_0 + 1, \cdots, N_x, \\ h_i &= 2(1 - 2\sigma_x) N_x^{-1}, && \text{for } i = i_0 + 1, \cdots, N_x - i_0, \end{aligned}$$

where $h_i = x_i - x_{i-1}$.

In the $y$-direction, we follow the same way above by dividing the interval $[0, 1]$ into the subintervals

$$[0, \sigma_y], \qquad [\sigma_y, 1-\sigma_y], \qquad [1-\sigma_y, 1].$$

Uniform meshes are then used on each subinterval, with $N_y/4$ points on each of $[0, \sigma_y]$ and $[1-\sigma_y, 1]$, and $N_y/2$ points on $[\sigma_y, 1-\sigma_y]$. Here $\sigma_y$ is defined by $\sigma_y = \min\{1/4, 2\alpha^{-1}\varepsilon \ln N_y\}$. More explicitly, we have

$$0 = y_0 < y_1 < \cdots < y_{j_0} < \cdots < y_{N_y - j_0} < \cdots < y_{N_y} = 1,$$

with $j_0 = N_y/4$, $y_{j_0} = \sigma_y$, $y_{N_y-j_0} = 1 - \sigma_y$, and

$$\begin{aligned} k_j &= 4\sigma_y N_y^{-1}, && \text{for } j = 1, \dots, j_0, \quad N_y - j_0 + 1, \dots, N_y, \\ k_j &= 2(1 - 2\sigma_y) N_y^{-1}, && \text{for } j = j_0 + 1, \dots, N_y - j_0, \end{aligned}$$

where $k_j = y_j - y_{j-1}$.

We shall assume that

$$\sigma_x = 2\alpha^{-1}\varepsilon \ln N_x, \qquad \sigma_y = 2\alpha^{-1}\varepsilon \ln N_y.$$

Otherwise $\varepsilon \geq \max(\alpha/8 \ln N_x, \alpha/8 \ln N_y)$, which is not a singularly perturbed problem. Then the problem can be analyzed in the classical way, which is not our interest here.

Let $I_i = [x_{i-1}, x_i]$, $I = [0, 1]$, $\tilde{I}_i = I_i \times I$, $h = \max_{1 \leq i \leq N_x} h_i$ , $K_j = [y_{j-1}, y_j]$, $\tilde{K}_j = I \times K_j$, $k = \max_{1 \leq j \leq N_y} k_j$ and $\|\cdot\|_\infty$ be the $L^\infty$ norm. We will use different subscripts to distinguish the norm on the corresponding subdomain.

The weak formulation of (1) is: find $u \in H_0^1(\Omega)$ such that

$$B(u, v) \equiv \left(\varepsilon^2 u_x, v_x\right) + \left(\varepsilon^2 u_y, v_y\right) + (au, v) = (f, v), \qquad \forall\, v \in H_0^1(\Omega), \tag{3}$$

where $(\cdot, \cdot)$ denote the usual $L^2(\Omega)$ inner product and $H_0^1(\Omega)$ is the usual Sobolev space [27–30].

Denote the weighted energy norm

$$|||v||| \equiv \left\{\varepsilon^2 \|v_x\|^2 + \varepsilon^2 \|v_y\|^2 + \|v\|^2\right\}^{1/2}, \qquad \forall\, v \in H_0^1(\Omega),$$

where $\|\cdot\|$ denote the usual $L^2$-norm.

Note that

$$\begin{aligned}B(v,v) &= \varepsilon^2\|v_x\|^2 + \varepsilon^2\|v_y\|^2 + (av,v)\\ &\geq \min\left(1,\alpha^2\right)\left(\varepsilon^2\|v_x\|^2 + \varepsilon^2\|v_y\|^2 + \|v\|^2\right)\\ &\geq \min\left(1,\alpha^2\right)|||v|||^2\end{aligned}$$

and by Cauchy-Schwarz inequality

$$\begin{aligned}B(v,w) &\leq \varepsilon^2\|v_x\|\ \|w_x\| + \varepsilon^2\|v_y\|\|w_y\| + \left(\max_{(x,y)\in\overline{\Omega}} a\right)\|v\|\ \|w\|\\ &\leq |||v|||\ |||w||| + |||v|||\ |||w||| + \left(\max_{(x,y)\in\overline{\Omega}} a\right)|||v|||\ |||w|||\\ &= \left(2 + \max_{(x,y)\in\overline{\Omega}} a\right)|||v|||\ |||w|||.\end{aligned}$$

Note that the mapping $v \to (f,v)$ is a bounded functional on $H_0^1$. Combining this fact with the above two inequalities, the Lax-Milgram Lemma [27–29] tells us that (3) has a unique solution $u(x,y)$ in $H_0^1(\Omega)$.

Let $S_h(\Omega)$ be the ordinary bilinear finite element space [31]. We seek the finite element solution $u^h \in S_h$ such that

$$\overline{B}\left(u^h,v\right) \equiv \left(\varepsilon^2 u_x^h, v_x\right) + \left(u_y^h, v_y\right) + \left(\overline{a}u^h, v\right) = \left(\overline{f}, v\right), \qquad \forall\, v \in S_h, \tag{4}$$

where $\overline{a}$ and $\overline{f}$ denote some piecewise polynomial approximation of $a$ and $f$, respectively, such that

$$\|(\overline{a}-a)\|_{\infty,\Omega} \leq C\left(h^p + k^p\right), \tag{5}$$

and

$$\left\|\left(\overline{f}-f\right)\right\|_{\infty,\Omega} \leq C\left(h^p + k^p\right), \tag{6}$$

where $p$ is the approximation order.

Let $\Pi w \equiv \sum_{i=0}^{N_x}\sum_{j=0}^{N_y} w_{ij} l_i(x) l_j(y)$ be the standard bilinear interpolate of $w$, where $\Pi_x$ and $\Pi_y$ are the interpolates in $x$- and $y$-directions, respectively. Here $l_i(x)$ is the well-known "hat" function [27,31].

Let us recall some results in [31] we will use in the next section.

LEMMA 4.1. *(See [31, Theorem 2.1].)* $\Pi w = \Pi_x\Pi_y w = \Pi_y\Pi_x w$.

LEMMA 4.2. *(See [31, Theorem 2.6].)* $\|w - \Pi_x w\|_{\infty,\bar{I}_i} \leq 1/8h_i^2\|w_{xx}\|_{\infty,\bar{I}_i}$.

LEMMA 4.3. *(See [31, Lemma 2.1].)*

$$\begin{aligned}&\|\Pi_x u\|_{\infty,\bar{I}_i} \leq \max_{y\in I}(|u(x_{i-1},y)|, |u(x_i,y)|),\\ &\|\Pi_x u\|_{\infty,\Omega} \leq \|u\|_{\infty,\Omega}.\end{aligned}$$

The same results in Lemmas 4.2 and 4.3 hold true for interpolate $\Pi_y$ in $y$-direction.

## 5. MAIN RESULTS

Let us first prove some error estimates for the solution $u$ of (1),(2).

LEMMA 5.1. *For the solution $u$ of (1),(2) and any integer $n \geq 0$, we have*

$$\text{(I)} \qquad \|u - \Pi_x u\|_{\infty,\tilde{I}_i} \leq C\left(N_x^{-2}\ln^2 N_x + \varepsilon^{2n+1}\right), \qquad \forall i = 1,\dots,N_x, \tag{7}$$

$$\text{(II)} \qquad \|u - \Pi_y u\|_{\infty,\tilde{K}_j} \leq C\left(N_y^{-2}\ln^2 N_y + \varepsilon^{2n+1}\right), \qquad \forall j = 1,\dots,N_y. \tag{8}$$

PROOF. First, for $i = 1,\dots,i_0,\ N_x - i_0 + 1,\dots,N_x$, by Lemma 3.5 and Lemma 4.2, we have

$$\begin{aligned}\|u - \Pi_x u\|_{\infty,\tilde{I}_i} \leq Ch_i^2 \|u_{xx}\|_{\infty,\tilde{I}_i} &\leq Ch_i^2 \max_{x\in I_i}\left(1 + \varepsilon^{-2}e^{-\alpha x/\varepsilon} + \varepsilon^{-2}e^{-\alpha(1-x)/\varepsilon}\right)\\ &\leq Ch_i^2\left(1+\varepsilon^{-2}\right) \leq CN_x^{-2}\ln^2 N_x,\end{aligned}$$

since $h_i = 4\sigma_x/N_x$ in this case. Hence (7) is true in this case.

Second, for $i = i_0+1,\dots,N_x - i_0$, in this case $x \in [\sigma_x, 1-\sigma_x]$. Use Theorem 2.1 for $n \geq 0$, we can write $\Pi_x u$ in the form

$$\Pi_x u = \Pi_x U_{2n} + \Pi_x V_{2n} + \Pi_x W_{2n} + \Pi_x \overline{V}_{2n} + \Pi_x \overline{W}_{2n} + \Pi_x\left(\sum_{l=1}^{4} Z_{2n}^l\right) + \Pi_x R_{2n},$$

where $\Pi_x U_{2n}$, $\Pi_x V_{2n}$, $\Pi_x W_{2n}$, $\Pi_x \overline{V}_{2n}$, $\Pi_x \overline{W}_{2n}$, $\Pi_x(\sum_{l=1}^4 Z_{2n}^l)$, and $\Pi_x R_{2n}$ denote the linear interpolation in $x$-direction to $U_{2n}$, $V_{2n}$, $W_{2n}$, $\overline{V}_{2n}$, $\overline{W}_{2n}$, $\sum_{l=1}^4 Z_{2n}^l$, and $R_{2n}$, respectively.

Note that $U_{2n}(x,y) = \sum_{i=0}^{2n} \varepsilon^i u_i(x,y)$ and $u_i(x,y)$ is independent of $\varepsilon$, we have

$$\|U_{2n} - \Pi_x U_{2n}\|_{\infty,\tilde{I}_i} \leq Ch_i^2 \|(U_{2n})xx\|_{\infty,\tilde{I}_i} \leq CN_x^{-2},$$

where in the last step we use the fact that $N_x^{-1} \leq h_i \leq 2N_x^{-1}$ for $i = i_0+1,\dots,N_x - i_0$.

By Lemma 2.1 and Lemma 4.3,

$$\left\|V_{2n}\left(x,\frac{y}{\varepsilon}\right) - \Pi_x V_{2n}\left(x,\frac{y}{\varepsilon}\right)\right\|_{\infty,\tilde{I}_i} \leq Ch_i^2 \|(V_{2n})_{xx}\|_{\infty,\tilde{I}_i} \leq CN_x^{-2},$$

$$\left\|\overline{V}_{2n}\left(x,\frac{(1-y)}{\varepsilon}\right) - \Pi_x \overline{V}_{2n}\left(x,\frac{(1-y)}{\varepsilon}\right)\right\|_{\infty,\tilde{I}_i} \leq Ch_i^2 \left\|\left(\overline{V}_{2n}\right)_{xx}\right\|_{\infty,\tilde{I}_i} \leq CN_x^{-2},$$

$$\begin{aligned}\left\|W_{2n}\left(\frac{x}{\varepsilon},y\right) - \Pi_x W_{2n}\left(\frac{x}{\varepsilon},y\right)\right\|_{\infty,\tilde{I}_i} &\leq 2\left\|W_{2n}\left(\frac{x}{\varepsilon},y\right)\right\|_{\infty,\tilde{I}_i}\\ &\leq Ce^{-\alpha x_{i-1}/\varepsilon} \leq Ce^{-\alpha\sigma_x/\varepsilon} = CN_x^{-2},\end{aligned}$$

$$\begin{aligned}\left\|\overline{W}_{2n}\left(\frac{(1-x)}{\varepsilon},y\right) - \Pi_x \overline{W}_{2n}\left(\frac{(1-x)}{\varepsilon},y\right)\right\|_{\infty,\tilde{I}_i} &\leq 2\left\|\overline{W}_{2n}\left(\frac{(1-x)}{\varepsilon},y\right)\right\|_{\infty,\tilde{I}_i}\\ &\leq Ce^{-\alpha(1-x_i)/\varepsilon} \leq Ce^{-\alpha\sigma_x/\varepsilon} = CN_x^{-2},\end{aligned}$$

Similarly by Lemma 2.1 and Lemma 4.3, we have

$$\begin{aligned}\left\|Z_{2n}^1\left(\frac{x}{\varepsilon},\frac{y}{\varepsilon}\right) - \Pi_x Z_{2n}^1\left(\frac{x}{\varepsilon},\frac{y}{\varepsilon}\right)\right\|_{\infty,\tilde{I}_i} &\leq 2\left\|Z_{2n}^1\left(\frac{x}{\varepsilon},\frac{y}{\varepsilon}\right)\right\|_{\infty,\tilde{I}_i} \leq Ce^{-\alpha x_{i-1}/\varepsilon}\\ &\leq Ce^{-\alpha\sigma_x/\varepsilon} = CN_x^{-2},\end{aligned}$$

$$\begin{aligned}\left\|Z_{2n}^2\left(\frac{(1-x)}{\varepsilon},\frac{y}{\varepsilon}\right) - \Pi_x Z_{2n}^1\left(\frac{(1-x)}{\varepsilon},\frac{y}{\varepsilon}\right)\right\|_{\infty,\tilde{I}_i} &\leq 2\left\|Z_{2n}^2\left(\frac{(1-x)}{\varepsilon},\frac{y}{\varepsilon}\right)\right\|_{\infty,\tilde{I}_i}\\ &\leq Ce^{-\alpha(1-x_i)/\varepsilon} \leq Ce^{-\alpha\sigma_x/\varepsilon} = CN_x^{-2},\end{aligned}$$

$$\left\|Z_{2n}^3\left(\frac{x}{\varepsilon},\frac{(1-y)}{\varepsilon}\right)-\Pi_x Z_{2n}^3\left(\frac{x}{\varepsilon},\frac{(1-y)}{\varepsilon}\right)\right\|_{\infty,\bar{I}_i} \le 2\left\|Z_{2n}^3\left(\frac{x}{\varepsilon},\frac{(1-y)}{\varepsilon}\right)\right\|_{\infty,\bar{I}_i}$$
$$\le Ce^{-\alpha x_{i-1}/\varepsilon} \le Ce^{-\alpha\sigma_x/\varepsilon} = CN_x^{-2},$$

$$\left\|Z_{2n}^4\left(\frac{(1-x)}{\varepsilon},\frac{(1-y)}{\varepsilon}\right)-\Pi_x Z_{2n}^4\left(\frac{(1-x)}{\varepsilon},\frac{(1-y)}{\varepsilon}\right)\right\|_{\infty,\bar{I}_i} \le 2\left\|Z_{2n}^4\left(\frac{(1-x)}{\varepsilon},\frac{(1-y)}{\varepsilon}\right)\right\|_{\infty,\bar{I}_i}$$
$$\le Ce^{-\alpha(1-x_i)/\varepsilon} \le Ce^{-\alpha\sigma_x/\varepsilon} = CN_x^{-2},$$

and

$$\|R_{2n}(x,y) - \Pi_x R_{2n}(x,y)\|_{\infty,\bar{I}_i} \le 2\|R_{2n}\|_{\infty,\bar{I}_i} \le C\varepsilon^{2n+1},$$

which concludes the proof of (I).

Similarly, we can prove (II) in the same way by symmetry.

Then we have the following lemma.

LEMMA 5.2. *For the solution $u$ of (1),(2) and any integer $n \ge 0$, we have*

$$\|u - \Pi u\|_{\infty,\Omega} \le C\left(N_x^{-2}\ln^2 N_x + N_y^{-2}\ln^2 N_y + \varepsilon^{2n+1}\right).$$

PROOF. Using Lemmas 4.1, 4.3, and 5.1, we have

$$\|u - \Pi u\|_{\infty,\Omega} \le \|u - \Pi_x u\|_{\infty,\Omega} + \|\Pi_x(u - \Pi_y u)\|_{\infty,\Omega} \tag{9}$$
$$\le \|u - \Pi_x u\|_{\infty,\Omega} + \|u - \Pi_y u\|_{\infty,\Omega} \tag{10}$$
$$\le \max_{1\le i\le N_x} \|u - \Pi_x u\|_{\infty,\bar{I}_i} + \max_{1\le j\le N_y} \|u - \Pi_y u\|_{\infty,\bar{K}_j} \tag{11}$$
$$\le C\left(N_x^{-2}\ln^2 N_x + N_y^{-2}\ln^2 N_y + \varepsilon^{2n+1}\right). \tag{12}$$

THEOREM 5.1. *Let $u_h$ be the finite element solution of (4) and $u$ be the solution of (1),(2). Assume $\overline{a}$ and $\overline{f}$ satisfy (5),(6), then for any integer $n \ge 0$, we have*

$$\|u - u^h\| \le C(1 + \varepsilon N_x + \varepsilon N_y)\left(N_x^{-2}\ln^2 N_x + N_y^{-2}\ln^2 N_y + \varepsilon^{2n+1}\right) + C\left(h^p + k^p\right).$$

PROOF. Note that

$$C_1|||\Pi u - u^h|||^2 \le \overline{B}\left(\Pi u - u^h, \Pi u - u^h\right) \tag{13}$$
$$= \overline{B}\left(\Pi u - u, \Pi u - u^h\right) + \overline{B}\left(u - u^h, \Pi u - u^h\right). \tag{14}$$

Let $\chi = \Pi u - u^h$, then from (4) we have

$$\overline{B}\left(\Pi u - u, \Pi u - u^h\right) = \varepsilon^2((\Pi u - u)_x, \chi_x) + \varepsilon^2((\Pi u - u)_y, \chi_y) + (\overline{a}(\Pi u - u), \chi). \tag{15}$$

Integrating by parts, we obtain

$$\begin{aligned}
\varepsilon^2((\Pi u - u)_x, \chi_x) &= \sum_{1\le i\le N_x, 1\le j\le N_y} \int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{y_j} \varepsilon^2(\Pi u - u)_x \chi_x \,dx\,dy \\
&= \sum_{1\le i\le N_x, 1\le j\le N_y} \int_{y_{j-1}}^{y_j} \varepsilon^2(\Pi u - u)|_{x=x_{i-1}}^{x=x_i} \chi_x \,dy \\
&\le \sum_{1\le i\le N_x, 1\le j\le N_y} \int_{y_{j-1}}^{y_j} |\varepsilon\chi_x| \,dy \cdot \varepsilon||\Pi u - u||_{\infty,\Omega} \\
&= \sum_{1\le i\le N_x} \int_0^1 |\varepsilon\chi_x| \,dy \cdot \varepsilon||\Pi u - u||_{\infty,\Omega} \\
&= \sum_{1\le i\le N_x} \int_0^1\int_0^1 |\varepsilon\chi_x| \,dy\,dx \cdot \varepsilon||\Pi u - u||_{\infty,\Omega}, \text{ since } |\varepsilon\chi_x| \text{ independent of } x, \\
&= \varepsilon N_x||\Pi u - u||_{\infty,\Omega} \cdot \int_0^1\int_0^1 |\varepsilon\chi_x| \,dy\,dx, \\
&\le C\varepsilon N_x\left(N_x^2\ln^2 N_x + N_y^2\ln^2 N_y + \varepsilon^{2n+1}\right)\|\varepsilon\chi_x\|, \qquad \text{by Lemma 5.2.}
\end{aligned}$$

Similarly, we have

$$\varepsilon^2((\Pi u - u)_y, \chi_y) \le C\varepsilon N_y \left(N_x^2 \ln^2 N_x + N_y^2 \ln^2 N_y + \varepsilon^{2n+1}\right) \|\varepsilon \chi_y\|. \tag{16}$$

Also note that

$$(\bar{a}(\Pi u - u), \chi) \le C\|\bar{a}\|_{\infty,\Omega}\|\Pi u - u\| \, \|\chi\| \le C\|\Pi u - u\|_{\infty,\Omega}\|\chi\| \tag{17}$$

$$\le C\left(N_x^{-2} \ln^2 N_x + N_y^{-2} \ln^2 N_y + \varepsilon^{2n+1}\right) \left\|\Pi u - u^h\right\|, \tag{18}$$

where in the last step we used Lemma 5.2. On the other hand,

$$\overline{B}\left(u - u^h, \Pi u - u^h\right) = \left(\overline{B} - B\right)\left(u, \Pi u - u^h\right) + B\left(u, \Pi u - u^h\right) - \overline{B}\left(u^h, \Pi u - u^h\right) \tag{19}$$

$$= \left((\bar{a} - a)u, \Pi u - u^h\right) + \left(f - \bar{f}, \Pi u - u^h\right) \tag{20}$$

$$\le C\left(\|\bar{a} - a\|_{\infty,\Omega} + \left\|f - \bar{f}\right\|_{\infty,\Omega}\right) \left\|\Pi u - u^h\right\|. \tag{21}$$

Using (5), (6), (13)–(21), and Lemma 4.2, we have

$$\left|\left|\left|\Pi u - u^h\right|\right|\right| \le C(1 + \varepsilon N_x + \varepsilon N_y)\left(N_x^{-2} \ln^2 N_x + N_y^{-2} \ln^2 N_y + \varepsilon^{2n+1}\right) + C\left(h^p + k^p\right).$$

Therefore, combining this with Lemma 4.2, we obtain

$$\left\|u - u^h\right\| \le \|u - \Pi u\| + \left\|\Pi u - u^h\right\| \le \|u - \Pi u\|_{\infty,\Omega} + \left|\left|\left|\Pi u - u^h\right|\right|\right| \tag{22}$$

$$\le C(1 + \varepsilon N_x + \varepsilon N_y)\left(N_x^{-2} \ln^2 N_x + N_y^{-2} \ln^2 N_y + \varepsilon^{2n+1}\right) + C\left(h^p + k^p\right) \tag{23}$$

which concludes our proof.

Since we are considering singularly perturbed problems, the parameter $\varepsilon$ is usually very small. Without loss of generality, we can assume $\varepsilon \le \max(N_x^{-1}, N_y^{-1})$. Then we obtain the following quasi-optimal uniform convergence result.

COROLLARY. *Let $u_h$ be the finite element solution of (9) and $u$ be the solution of (1),(2). Let $\bar{a}$ and $\bar{f}$ be the bilinear interpolation of $a$ and $f$, respectively. Then we have*

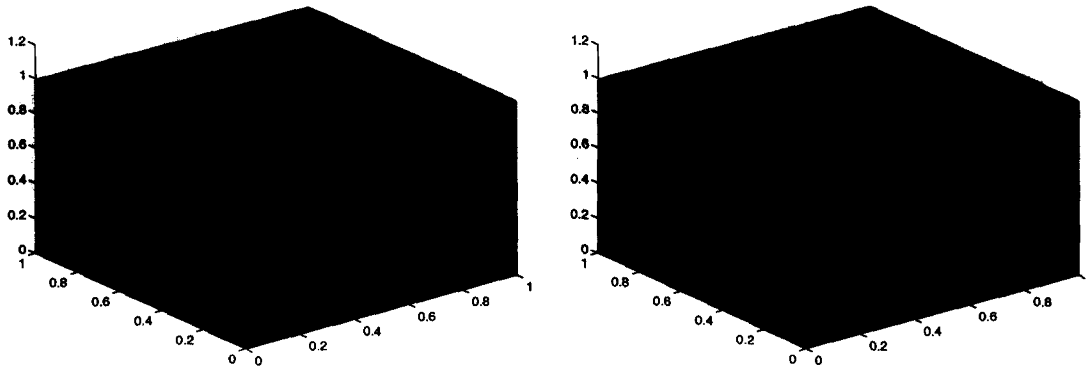
$$\left\|u - u^h\right\| \le C\left(N_x^{-2} \ln^2 N_x + N_y^{-2} \ln^2 N_y\right).$$

PROOF. Since $\bar{a}$ and $\bar{f}$ are the bilinear interpolates of $a$ and $f$, respectively, we have $p = 2$ in (5),(6). We can choose $n$ large enough such that $\varepsilon^{2n+1} \le \max(N_x^{-2} \ln^2 N_x, N_y^{-2} \ln^2 N_y)$ is satisfied. Using Theorem 5.1 concludes our proof.

REMARK 5.1. When $f$ depends on $\varepsilon$ and satisfies the assumptions in [32, p. 128], we can see that the above results still hold true by carrying out a similar proof.

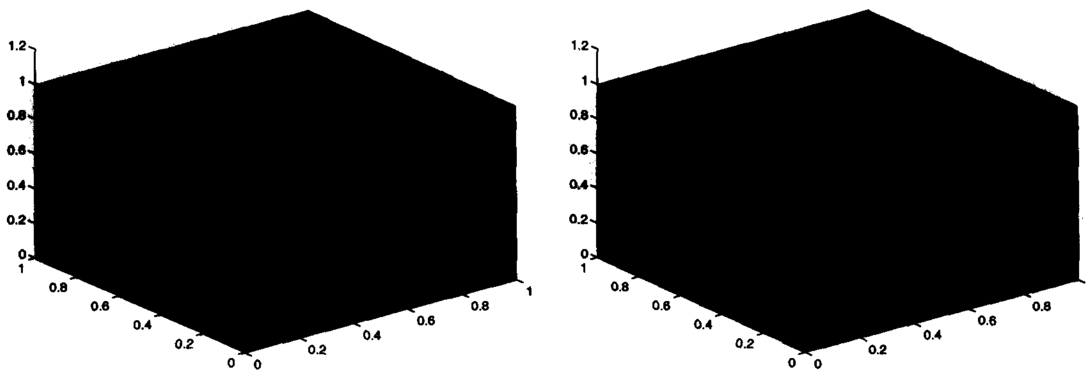
Table 1. Errors in $L^2$-norm.

| | $N$ | | | |
|---|---|---|---|---|
| $\varepsilon$ | 12 | 24 | 48 | 96 |
| 1.0D−02 | 4.08804D−02 | 1.13088D−02 | 2.48833D−03 | 4.14854D−04 |
| 1.0D−03 | 5.01689D−02 | 1.68489D−02 | 5.59478D−03 | 1.79468D−03 |
| 1.0D−04 | 5.12377D−02 | 1.75666D−02 | 6.07620D−03 | 2.10844D−03 |
| 1.0D−05 | 5.13461D−02 | 1.76404D−02 | 6.12728D−03 | 2.14388D−03 |
| 1.0D−06 | 5.13569D−02 | 1.76478D−02 | 6.13242D−03 | 2.14747D−03 |
| 1.0D−07 | 5.13580D−02 | 1.76486D−02 | 6.13293D−03 | 2.14783D−03 |
| 1.0D−08 | 5.13581D−02 | 1.76487D−02 | 6.13298D−03 | 2.14787D−03 |
| $N^{-2} \ln^2 N$ | 4.29D−02 | 1.75D−02 | 6.5D−03 | 2.3D−03 |



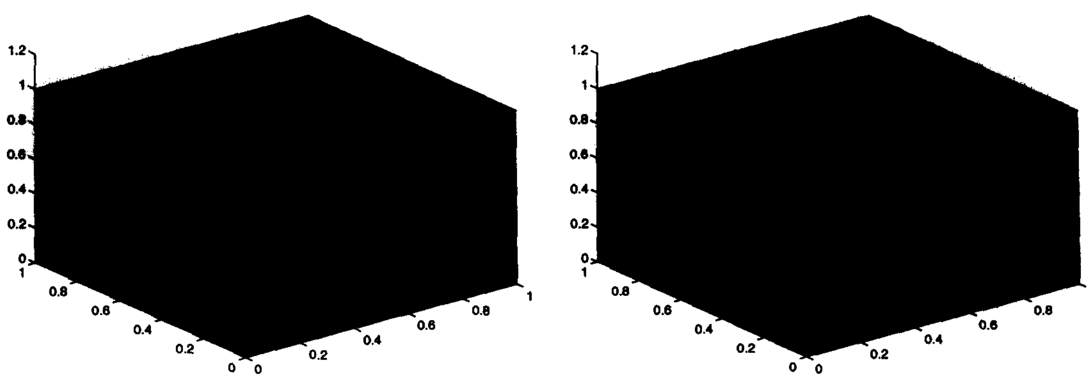

(a) $N = 24$. (b) $N = 48$.

Figure 1. Computed FEM solution for $\varepsilon = 1.0D - 03$.




(a) $N = 24$. (b) $N = 48$.

Figure 2. Computed FEM solution for $\varepsilon = 1.0D - 05$.

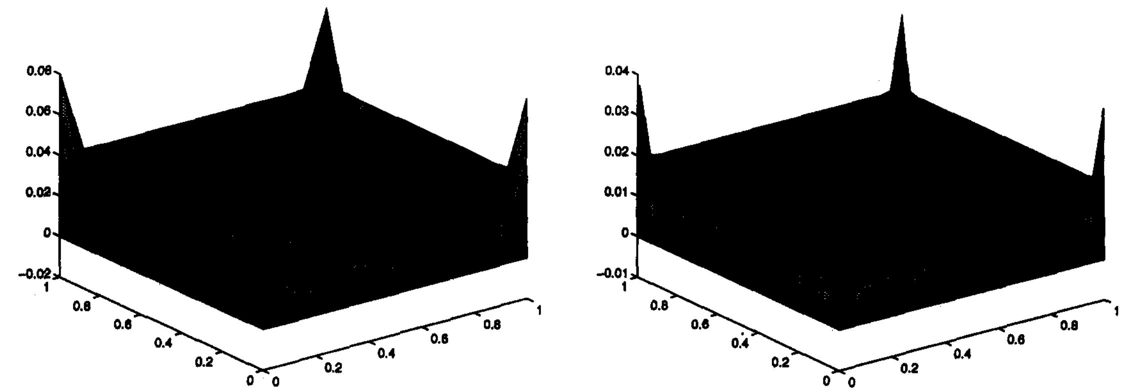



(a) $N = 24$. (b) $N = 48$.

Figure 3. Computed FEM solution for $\varepsilon = 1.0D - 07$.

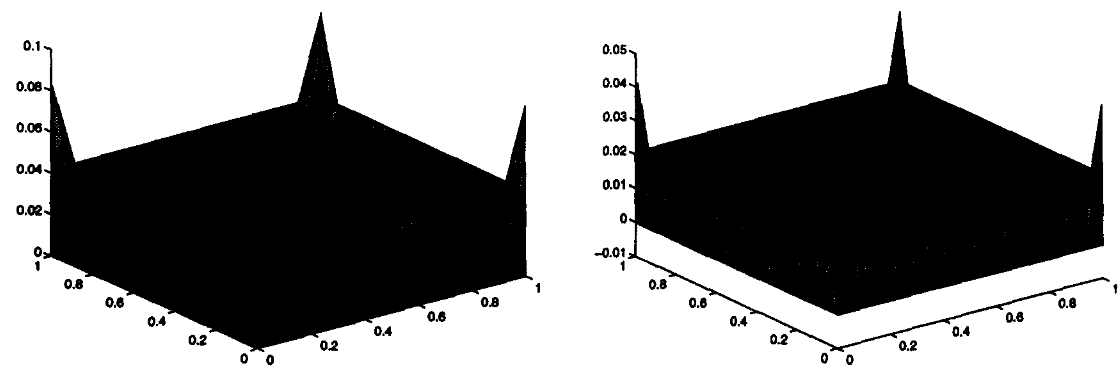
## 6. NUMERICAL RESULTS

To see how our method performs, we tested here the example problem (1),(2) where $a = 2$ and $f$ is chosen such that the solution of (1),(2) is



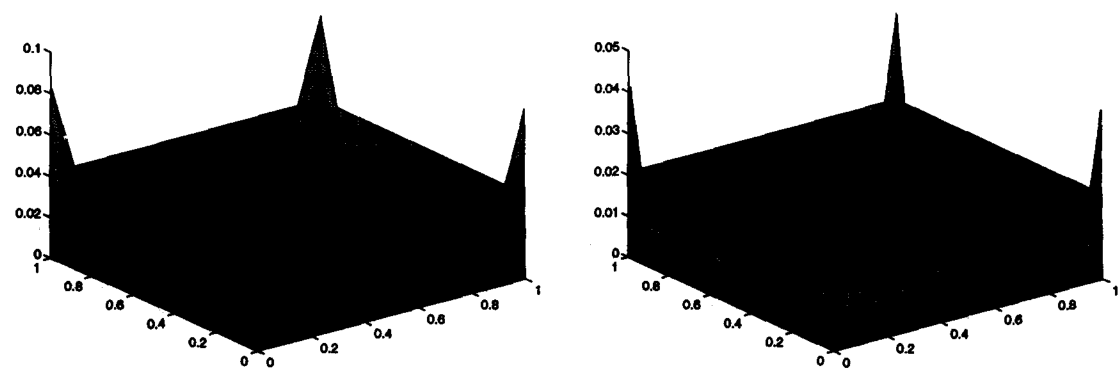
(a) $N = 24$. (b) $N = 48$.

Figure 4. Pointwise error $u_h - u$ for $\varepsilon = 1.0D - 03$.



(a) $N = 24$. (b) $N = 48$.

Figure 5. Pointwise error $u_h - u$ for $\varepsilon = 1.0D - 05$.



(a) $N = 24$. (b) $N = 48$.

Figure 6. Pointwise error $u_h - u$ for $\varepsilon = 1.0D - 07$.

$$u(x,y) = \left(1 - \frac{e^{-x/\varepsilon} + e^{-(1-x)/\varepsilon}}{1 + e^{-1/\varepsilon}}\right)\left(1 - \frac{e^{-y/\varepsilon} + e^{-(1-y)/\varepsilon}}{1 + e^{-1/\varepsilon}}\right).$$

This $u$ has the typical boundary layer behaviour. Since the exact solution is known, we can accurately measure the solution errors. We choose a bilinear interpolation $\Pi f$ of $f$ as $\overline{f}$ and $N_x = N_y = N$. All our computations are carried on IBM RS/6000 in double precision. The numerical results of our experiments for values of $\varepsilon$ varying in the interval $10^{-8} - 10^{-2}$ and various mesh resolutiones $N \in [12, 96]$ are shown in Table 1. They display an uniform convergence (i.e., independent of $\varepsilon$) in $L^2$-norm.

The computed solutions $u_h$ were plotted in Figures 1–3 for $\varepsilon = 10^{-3}$, $10^{-5}$, $10^{-7}$, and $N = 24, 48$. The pointwise errors were plotted in Figures 4–6 for the same $\varepsilon$ and $N$. From these figures, we see that our method solves this type of problem quite well. The boundary layers are much sharper and no oscillations are observed near the boundary layers. These phenomena were also observed by Madden *et al.* [20] for FEM on Shishkin meshes for convection-diffusion problems. But they did not present any theoretical analysis.

Table 2. Convergence rates $R_\varepsilon^N$ in $L^2$-norm.

| | $N$ | | |
|---|---|---|---|
| $\varepsilon$ | 12 | 24 | 48 |
| 1.0D−02 | 2.8741 | 3.0533 | 3.3901 |
| 1.0D−03 | 2.4403 | 2.2234 | 2.1516 |
| 1.0D−04 | 2.3942 | 2.1410 | 2.0029 |
| 1.0D−05 | 2.3895 | 2.1326 | 1.9872 |
| 1.0D−06 | 2.3891 | 2.1317 | 1.9857 |
| 1.0D−07 | 2.3890 | 2.1317 | 1.9855 |
| 1.0D−08 | 2.3890 | 2.1316 | 1.9855 |

To see more accurately the convergence rate, let $e_\varepsilon^N$ be the $L^2$-norm between the exact solution $u(x, y)$ and the computed solution $u^h(x, y)$. The computed convergence rate can be obtained by

$$R_\varepsilon^N = \frac{\left(\ln e_\varepsilon^{2N} - \ln e_\varepsilon^N\right)}{\ln\left(\ln(2N)/2\ln N\right)}.$$

The results are given in Table 2. From Table 2, we see that $u^h(x, y)$ approximates $u(x, y)$ with an accuracy order of $O(N^{-2}\ln^2 N)$ in $L^2$-norm, which is the same as obtained by our theoretical analysis.

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