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Edited by

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CONSERVATION LAWS IN FLUID DYNAMICS AND THE ENFORCEMENT OF THEIR PRESERVATION IN NUMERICAL DISCRETIZATIONS

I. M. Navon

1. INTRODUCTION

The existence of conservation laws for fluid dynamics problems has been recognized to be of vital importance to the understanding of basic physi= cal and mathematical properties of problems of interest. Almost all the equations describing fluid dynamics problems can be interpreted as the laws of conservation of mass, momentum and energy. In this short lecture note we will adopt two different approaches which essentially characterize different lines of thought as far as conservation laws are concerned. The first is the 'theoretical' approach whereby one considers what is a conservation law and the problem of obtaining conservation laws in the sense of physics from a system of equations as an integrability problem on a manifold (Eiseman and Stone 1980 [1]).

In this approach we are also concerned with the preservation of conserva= tion laws under coordinate transformations. One then might ask how many conservation laws are contained in a system of partial differential equa= tions describing some fluid dynamic phenomenon. This brings us to the Korteweg-de Vries equations which possess an infinity of conservation laws and to other similar equations. This can be connected with other types of conservation laws recently found for the KdV equation by Wahlquist and Estabrook [2]. For another excellent survey on this topic see Manin [3]. This topic is also related to the inverse scattering problem.

For the discretized non-linear partial differential equations of fluid dynamics the author has preferred to bring examples from numerical weather prediction and the accent was put mainly on finite difference schemes. Here conservative spatial finite difference discretizations were frequently used in order to reduce the tendency towards non-linear instability (Gary [4], Kalnay de Rivas et al [5]).

There is still a controversy if the use of a conservative spatial approxi= mation will totally avoid non-linear instability (Gary [4]). However in atmospheric sciences and numerical weather prediction it is common ex= perience that enstrophy conserving schemes and quadratically or energy conserving schemes provide a more accurate approximation to the spectrum and avoid the unbounded growth associated with catastrophic non-linear instability.

It should be stressed that in any numerical model, *the finite resolution imposes an artificial 'wall' at the short-wave end of the spectrum*, in= ducing an excessive accumulation of energy in the shortest waves (Kalnay de Rivas et al [5]). This problem is worst in non-conservative schemes but it appears even in alias-free, energy and enstrophy conserving spectral models. This justifies using besides the conservative finite-difference schemes some parametrization of the unresolved subgrid eddies to withdraw energy from the smallest resolved scales (Kalnay de Rivas et al [5]), (Gary [4]).

In this lecture-note we have stressed only some aspects of finite diffe= rence conserving schemes. A survey of 'a posteriori' methods for enforcing conservation in discretized finite difference models is also provided. Two small sections are dedicated to the conservative properties of spectral and finite element discretization methods.

Finally it should be mentioned that we are far from having exhausted the fluid-dynamics conservation laws (see Mobbs [6]).

We have also not treated systems of non-linear hyperbolic partial differential equations in conservation law form (see Lax [7]), the entropy condition, shock-waves, rarefaction-waves and contact discontinuities. The interested reader is referred to Liu [8], [9], Glium, [10] and Glium and Lax [11].

1. THE THEORETICAL APPROACH TO CONSERVATION LAWS

1.1 The Basic Equations

Definition 1. A conservative law on a manifold is defined exactly as any exact differential 1-form θ whose image $h\theta$ under a linear operator h, is also exact.

Definition 2. The rate of change of a quantity U in a volume M is classically expressed as the negative sum of fluxes F(U) of a quantity U across a boundary ∂M.

$$\frac{\partial}{\partial t}\int_M U dV = \int F(U) dS \tag{1}$$

where dV and dS are volume elements on M and ∂M respectively. Using Stokes theorem and the fact that M was arbitrarily chosen, we obtain finally the partial differential equation

$$\frac{\partial U}{\partial t} + \nabla \cdot F(U) = 0 \tag{2}$$

in *divergence form* or in conservation law form.

Example 1

The divergence form of a heat conducting viscous fluid is given by the continuity equation

$$\frac{d\rho}{dt} + \nabla \cdot (\rho V) = 0, \tag{3}$$

the energy equation

$$\frac{\partial E}{\partial t} + \nabla \cdot \{EV - K\nabla T + [\tau]\cdot V\} = 0 \tag{4}$$

K thermal conductivity

ρ density

T temperature

$\otimes$ the tensor product,

E total energy

V velocity vector

$[\tau]$ the stress tensor

and the momentum equation

$$\frac{\partial}{\partial t}(\rho V) + \nabla \cdot \{\rho V \otimes V + [\tau]\} = 0 \tag{5}$$

$$E = \rho(\mathcal{E} + \tfrac{1}{2}\|V\|^2) \tag{6}$$

where $\mathcal{E}$ is the internal energy, and E the total energy. The stress-tensor for a Newtonian fluid is:

$$[\tau] = (p - \lambda\nabla\cdot V)[I] - 2\mu[D] \tag{7}$$

where

p is pressure

λ,μ viscosity coefficients

[I] the identity dyadic

[D] the deformation tensor

$\otimes$ is the tensor product

For an ideal gas the equation of state is given by:

$$p = \rho RT \tag{8}$$

where

cp specific heat at constant pressure

R the gas constant

$\mathcal{E}$ internal energy = CvT

cv specific heat at constant volume

R = cp - cv

The conservation law form of the fluid dynamic equations is important for the numerical computation of certain flow fields. This is particularly true for flows with shock waves where in fact the Rankine-Hugoniot conditions can be satisfied by a mesh adjustment.

1.2 Fluid dynamic equations in curvilinear coordinates

Following the approach of Eiseman and Stone [1] let $x^1,\dots,x^n$ be fixed Cartesian coordinates and let $y^1,\dots,y^n$ be curvilinear coordinates. The local vector fields $\{\partial/\partial x^1,\dots,\partial/\partial x^n\}$ form the basis of R^n while $\{\partial/\partial y^1,\dots,\partial/\partial y^n\}$ are an alternate basis related to the first by

$$\partial/\partial y^i = (\partial x^\alpha/\partial y^i)(\partial/\partial x^\alpha) \qquad (9)$$

using the Einstein summation convention.

We assume the Jacobian $J = \det(\partial x^i/\partial y^j)$ to be different from 0 and we denote

$$\vec{e}_i = \partial/\partial y^i \ , \ i=1,2,\dots,n\,. \qquad (10)$$

A metric on the vector fields is an inner product $<\cdot,\cdot>$. By applying the metric to the basis elements $\vec{e}_i$ we get

$$g_{ij} = <e_i,e_j> = \frac{\partial x^\alpha}{\partial y^i}\frac{\partial x^\beta}{\partial y^j}<\frac{\partial}{\partial x^\alpha},\frac{\partial}{\partial y^\beta}>$$

$$= \frac{\partial x^\alpha}{\partial y^i}\frac{\partial x^\beta}{\partial y^j}\delta_{\alpha\beta} = \frac{\partial x^\alpha}{\partial y^i}\frac{\partial x^\alpha}{\partial y^j}, \text{ summation on } \alpha. \qquad (11)$$

The differential element of arc length ds can be obtained from the relationship:

$$(ds)^2 = \delta_{\sigma\beta}\, dx^\alpha\, dx^\beta = \delta_{\alpha\beta}\frac{\partial x^\alpha}{\partial y^\omega}\frac{\partial x^\beta}{\partial y^\sigma} dy^\omega\, \partial y^\sigma = g_{\omega\sigma}\, dy^\omega\, dy^\sigma. \qquad (12)$$

If A is the Jacobian transformation $\partial x^i/\partial y^j$ we have $g_{ij} = AA^t$; then

$$g \equiv \det(g_{ij}) = \det AA^t = (\det A)^2 = J^2 \qquad (13)$$

i.e. (g_{ij}) is non-singular if $J\neq 0$ i.e. the matrix g_{ij} has an inverse which will be denoted by (g_{ij}).

Following Eiseman and Stone [1], we note that to express the system (3)-(5) of fluid dynamic equations in curvilinear coordinates we need to define a dual basis $\{\vec{e}^1,\dots,\vec{e}^n\}$ by

$$<\vec{e}_i,\vec{e}^j> = \delta_i^j \qquad (14)$$

where

$$\delta_j^i = \delta_{ij} \qquad (15)$$

Then the gradient is defined as

$$\nabla = \vec{e}^k \otimes D_k \qquad (16)$$

and the divergence ∇ is defined by replacing the tensor product in ∇ by a dot product. To obtain the dot product one applies the metric to $\vec{e}^k$ and the first component of the tensor on the right.

Furthermore if $\sqrt{g} = |J|$ the absolute value of the Jacobian is independent of time, each equation can be multiplied by $\sqrt{g}$ and $\sqrt{g}$ can also be brought under the time derivative.

The continuity equation then becomes

$$\frac{\partial}{\partial t}(\rho\sqrt{g}) + \frac{\partial}{\partial y^i}(\rho v^i\sqrt{g}) = 0\ . \qquad (17)$$

The energy equation becomes

$$\frac{\partial}{\partial t}(E\sqrt{g}) + \frac{\partial}{\partial y^i}[(Ev^i - g^{ij}K\frac{\partial T}{\partial y^j} + g_{rj}\,\tau^{ij}\, v^r)\sqrt{g}] = 0 \qquad (18)$$

and the momentum equation is given by:

$$\{\frac{\partial}{\partial t}(\rho v^j\sqrt{g}) + \frac{\partial B^{ij}}{\partial y^i} + B^{ir}\ \Gamma^j_{ir}\}\ \vec{e}_j = 0 \qquad (19)$$

where

$$\Gamma^j_{ik} = \frac{g^{j\alpha}}{2} \{ \frac{\partial g_{\alpha i}}{\partial y^k} + \frac{\partial g_{\alpha k}}{\partial y^i} - \frac{\partial g_{ik}}{\partial y^\alpha} \} \quad (20)$$

is the Levi-Civita tensor, designed to be compatible with the metric

$$B^{ij} = (\rho v^i v^j + \tau^{ij})\sqrt{g} \quad (21)$$

while

$$[\tau] = \tau^{ij} \vec{e}_i \otimes \vec{e}_j = \{g^{ij}(p - \frac{\gamma}{\sqrt{g}} \frac{\partial}{\partial y^m}(v^m\sqrt{g}))$$

$$- \mu g^{im}(\frac{\partial v^j}{\partial y^m} + v^\ell \Gamma^j_{m\ell}) - \mu g^{jm}(\frac{\partial v^i}{\partial y^m} + v^\ell \Gamma^i_{m\ell})\} \vec{e}_i \otimes \vec{e}_j \ . \quad (22)$$

In the curvilinear coordinate system the continuity and energy equations remain in conservation law form *while the momentum equation misses this form due to the source-like term* $B^{ir} \Gamma^j_{ir}$. To transform it into a conser= vation law form one has to rewrite the momentum equations in a way that absorbs the source-like term. By using the work of Anderson, Preiser and Rubin [12] and implementing the method of integrating factors, one obtains momentum equations in conservative form:

$$\frac{\partial}{\partial t}(\rho v^s \sqrt{g} \frac{\partial x^m}{\partial x^s}) + \frac{\partial}{\partial y^j} (B^{ij} \frac{\partial x^m}{\partial y^i}) = 0 \qquad m=1,2,\ldots n \ . \quad (23)$$

The conservation law form of the fluid dynamic equations is important for the numerical solution of flow fields. In particular this is true for flows with shock-waves where the Rankine-Hugoniot conditions can be sa= tisfied by a mesh-adjustment.

1.3 Conservation law forms from Stoke's theorem and differential forms

In a region M of an M-dimensional Euclidean space a quantity U is con= served as a function of time if the rate of change of U in M is equal to the negative sum of the flux ω of quantity U across the boundary ∂M. Using Flanders [13] notation of exterior differential forms, the conservation of U is given by

$$\frac{\partial}{\partial t} \int_M U \otimes dV = - \int_{\delta M} \omega \quad (24)$$

for a positively oriented n-form dV as a volume element on M and for some (n-1) form ω which is some function of U that describes the U-flux through ∂M. If $x^1, x^2, \ldots, x^n$ are Cartesian coordinates ordered in such a way that $dV = dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n$ the flux can be expressed as

$$\omega = \sum_{i=1}^{n} (-1)^i F^i(U) \otimes dx^1 \wedge \ldots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \ \ldots \wedge dx^n \quad (25)$$

where F^i is the flux in direction x^i.

When U is a vector then F^i are vectors in the same space and ω is a (n-1) vector form. Using Stokes theorem we obtain

$$\int_M \frac{\partial U}{\partial t} \otimes dV = \frac{\partial}{\partial t} \int_M U \otimes dV = \int_{\delta M} \omega = \int_M d\omega = \int_M (\sum_{i=1}^{n} \frac{\partial F^i}{\partial x^i}) \otimes dV \ . \quad (26)$$

Remembering that the region M is arbitrary, we obtain the conservation law form

$$\frac{\partial U}{\partial t} + \sum_{i=1}^{n} \frac{\partial F^i(U)}{\partial x^i} = 0 \ . \quad (27)$$

An alternative definition of a conservation law is any pde in n indepen= dent variables $\{x^1,\ldots,x^n\}$ and p dependent variables $\{y^1,\ldots,y^p\}$ which has the form:

$$\sum_{i=1}^{n} \frac{\partial \psi^i}{\partial x^i} = 0 \quad (28)$$

where ψ^i are functions of $y^1,\ldots y^p$.

Example 2

The equations describing an inviscid ideal fluid in general curvilinear coordinates are given by the system

$$\frac{\partial}{\partial t}(\rho\sqrt{g}) + \frac{\partial}{\partial y^j}(\beta^s \frac{\partial y^j}{\partial x^s} \sqrt{g}) = 0 \quad (29)$$

$$\frac{\partial}{\partial t}(E\sqrt{g}) + \frac{\partial}{\partial y^j}\{[(E+p)\beta^s \frac{\partial y^j}{\partial x^s}]\sqrt{g}\} = 0 \qquad (30)$$

$$\frac{\partial}{\partial t}(\beta^\ell \sqrt{g}) + \frac{\partial}{\partial y^j}[(\frac{\beta^\ell \beta^s}{\rho} + \delta^{\ell s} p)\frac{\partial y^j}{\partial x^s}\sqrt{g}] = 0 \qquad (31)$$

where

$$p = (\gamma-1)[E - \tfrac{1}{2}(\sum_r \{\frac{\beta^r}{\rho}\}^2)\rho] \qquad (32)$$

and

$$\beta^m = \alpha^j \frac{\partial x^m}{\partial y^j}$$

α^j - are contravariant coefficients

$$\gamma = cp/cv \quad .$$

1.4 <u>Expression of a system of pde's in terms of conservation laws</u>

Consider the conversion of a system of the form

$$\frac{\partial}{\partial t}(u) + (h)\frac{\partial}{\partial x}(u) = 0 \qquad (33)$$

where (u) is a column vector

(h) is a square matrix with entries depending only on (u).

Can we find a suitable nonsingular matrix depending only on (u) to obtain a system of conservation laws.

$$\frac{\partial}{\partial t}U + \frac{\partial}{\partial x}V = 0 \qquad (34)$$

where U and V are column-vectors depending only on (u)?

Example 3

Let us take the equations of one dimensional time-dependent, nonisentropic, adiabatic fluid flow:

$$\frac{\partial}{\partial t}\begin{bmatrix}\rho\\u\\p\end{bmatrix} = \begin{bmatrix}u & \rho & 0\\0 & u & 1/\rho\\0 & \gamma p & u\end{bmatrix}\frac{\partial}{\partial x}\begin{bmatrix}\rho\\u\\p\end{bmatrix} = 0^{(*)} \qquad (35)$$

ρ, u, p, γ - are density, velocity, pressure and adiabatic constant res= pectively.

If S is entropy and $p=p(\rho,S)$ and one assumes $\gamma > 1$ then (35) is equivalent to the system

$$\frac{\partial}{\partial t}\begin{bmatrix}\rho\\u\\S\end{bmatrix} + \begin{bmatrix}u & \rho & 0\\c^2/\rho & u & p_S/\rho\\0 & 0 & u\end{bmatrix}\frac{\partial}{\partial x}\begin{bmatrix}\rho\\u\\S\end{bmatrix} = 0 \qquad (36)$$

where $p_\rho = \gamma p/\rho = c^2 > 0$.

Note that the eigenvalues of the 3×3 matrices which appear in (35) or (36) are distinct. These are u-c, u, u+c, and their corresponding eigen= vectors are $(0,\rho c^2,-c),(c^2,0,1)$ and $(0,\rho c^2,c)$. If $\rho > 0$, which is the physical case, we can premultiply the system (35) by the nonsingular matrix

$$\begin{bmatrix}1 & 0 & 0\\u & \rho & 0\\u^2 & 2\rho u & \frac{2}{\gamma-1}\end{bmatrix} \qquad (37)$$

to obtain the following system of conservation laws

$$\frac{\partial}{\partial t}\begin{bmatrix}\rho\\\rho u\\\rho u^2 + \frac{2p}{\gamma-1}\end{bmatrix} + \frac{\partial}{\partial x}\begin{bmatrix}\rho u\\\rho u^2+p\\\rho u^3 + \frac{2\gamma}{\gamma-1}pu\end{bmatrix} = 0 \,. \qquad (38)$$

1.5 Conservation laws and variational principles (Noether's theorem)

Let us take again the form

$$\sum_{i=1}^{n} \frac{\partial \psi^i}{\partial x^i} = 0 \ . \tag{39}$$

The functions ψ^i may themselves be in the form of derivatives. Let f be a function of y^j, $\frac{\partial y^j}{\partial x^i}$ and x^i, j=1,...,p and i=1,...,n.

Let

$$y^j_i = \frac{\partial y^j}{\partial x^i} \ , \ x = [x^i] \ . \tag{40}$$

We consider the n-fold integral $\int_R$ fdx where R is a closed bounded region and we assume that the integral is invariant under a family of transfor= mations with an arbitrary finite number of parameters. Further we suppose that f does not explicitly depend on the n independent variables x^i which can then be taken as parameters.

If we now look for a stationary value for $\int_R$ fdx we obtain p Euler varia= tional equations, which are:

$$f_{y^j} = \sum_{i=1}^{n} \frac{\partial}{\partial x^i} f_{y^j_i} = 0 \ . \tag{41}$$

Noether's theorem is a means of associating a conservation equation with an infinitesimal transformation having an action integral invariant.

In our notation Noether's result asserts that n linear combinations of (41) which have the form of conservation laws can be found. I.e., there exist functions ψ^i_k, $0 \le k \le n$ such that:

$$a^\alpha_k(f_{y^\alpha} - \sum_{i=1}^{n} \frac{\partial}{\partial x^i} f_{y^\alpha_i}) = \frac{\partial \psi^1_k}{\partial x^1} + \dots \frac{\partial \psi^n_k}{\partial x^n} = 0 \tag{42}$$

where a^j_k is an (n×p) matrix and the range of summation on α is from 1 to p.

Other relations with similar form are

$$\frac{\partial}{\partial x^i}(y^j_k) + \frac{\partial}{\partial x^k}(-y^j_i) = 0 \tag{43}$$

The number of conservation laws which appear in (43) depends in general on the group of transformations which fixes $\int_R$ fdx.

Example 4

We seek a stationary value for

$$J = \int_R \int (w_x^2 + w_y^2) dx \ dy \tag{44}$$

where R is the unit disc. Here we have n=2, p=1. The conservation laws we can obtain from (44) have the form

$$\frac{\partial}{\partial x}\begin{bmatrix} w_x \\ w_y \end{bmatrix} + \frac{\partial}{\partial y}\begin{bmatrix} w_y \\ -w_x \end{bmatrix} = 0 \ . \tag{45}$$

The first of the above relationships is the variational equation

$$w_{xx} + w_{yy} = \Delta w = 0 \ , \tag{46}$$

One observes that J is invariant under a one-parameter transformation group given by

$$\begin{aligned} x^* &= x(\cos\theta) + y(\sin\theta) \\ y^* &= -x(\sin\theta) + y(\cos\theta) \end{aligned} \tag{47}$$

That is

$$\int_R \int (w_x^2 + w_y^2) dx \ dy = \int_{R^*} \int (w_{x*}^2 + w_{y*}^2) dx^* \ dy^* \quad \text{where } R = R^* \ . \tag{48}$$

When the number of dependent variables is greater than two, the problem of obtaining conservation laws becomes more difficult. See Osborn [15].

For the existence of conservation laws on manifolds see Eiseman and Stone [1

Example 5

Conservative form of the time-dependent Navier-Stokes equations

The Navier-Stokes equations for plane-flows are written in cartesian coordinates (x,y) as follows.

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} = 0 \qquad (49)$$

The unicolumn matrice U,F and G have the following expressions:

$$U = [\rho, \rho u, \rho v, \rho E]^T \qquad (50)$$

while F and G are split into

$$(F,G) = \underset{\text{inviscid terms}}{(F_1,G_1)} - \underset{\text{viscous terms}}{(R,S)} \qquad (51)$$

or component-wise

$$\begin{aligned} F &= F_1(U) - R(W, \partial W/\partial x, \partial W/\partial y) \\ G &= G_1(U) - S(W, \partial W/\partial x, \partial W/\partial y) \end{aligned} \qquad (52)$$

$$W = \begin{bmatrix} u \\ v \\ e \end{bmatrix} \qquad (53)$$

$$p = (\gamma - 1)\rho e \qquad (54)$$

$$e = c_v T$$

$$F_1(U) = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \\ (\rho E + p)u \end{bmatrix}, \quad G_1(U) = \begin{bmatrix} \rho v \\ \rho u v \\ \rho v^2 + p \\ (\rho E + p)v \end{bmatrix} \qquad (55)$$

$$R = \begin{bmatrix} 0 \\ \tau_{xx} \\ \tau_{xy} \\ u\tau_{xx} + v\tau_{xy} + \frac{K}{c_v}\frac{\partial e}{\partial x} \end{bmatrix} \quad S = \begin{bmatrix} 0 \\ \tau_{yx} = \tau_{xy} \\ \tau_{yy} \\ u\tau_{xy} + v\tau_{yy} + \frac{K}{c_v}\frac{\partial e}{\partial y} \end{bmatrix} \qquad (56)$$

K - thermal conductivity

q - KgradT

Stress tensor

$$\begin{aligned} \tau_{xx} &= (\lambda + 2\mu)\frac{\partial u}{\partial x} + \lambda \frac{\partial v}{\partial y} \\ \tau_{xy} &= \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \\ \tau_{yy} &= \lambda \frac{\partial u}{\partial x} + (\lambda + 2\mu)\frac{\partial v}{\partial y} \end{aligned} \qquad (57)$$

λ, μ - viscosity coefficients

$\lambda = \lambda(T)$, $\mu = \mu(T)$, $K = K(T)$

$$3\lambda + 2\mu \geq 0, \ \mu \geq 0 \ . \qquad (58)$$

If one assumes local thermodynamic equilibrium one always makes use of the Stokes relation

$$3\lambda + 2\mu = 0 \qquad (59)$$

See also Viviand [16] and Vinokut [17].

1.6 The Korteweg de Vries Equation and its Conservation Laws

We consider the equation

$$u_t + uu_x + u_{xxx} = 0 \qquad (60)$$

and its generalizations.

In this context a conservation law associated with an equation such as (60) is expressed by an equation of the form

$$T_t + X_x = 0 \qquad (61)$$

where T the *conserved density* and -X, the flux of T, are functionals of u.

Definition. (Kruskal, Muira and Gardner [18])

Local conservation laws

If T is a *local* functional of u, i.e. if the value of T at any x depends only on the values of u in an arbitrarily small neighbourhood of x, then T is a *local conserved density*. If X is also local, then (61) is a *local conservation law*. If T is in particular a polynomial in u and its x derivatives are not dependent explicitly on x and t, then we call T a polynomial conserved density; if X is also such a polynomial we call (61) a polynomial conservation law. There is a close relationship between constants of motion and conservation laws.

For KdV equations (for a polynomial conservation law) T and X are each a finite sum of terms of the form $u_0^{a_0}, u_1^{a_1}, \ldots, u_\ell^{a_\ell}$ where

$$u_j = \frac{\partial^j u}{\partial x^j} \qquad (62)$$

and a_j are nonnegative integers.

The rank is defined as the sum of the number of factors a_j and half the number of x differentiations

$$r_1 = \sum_{j=0}^{\ell} (1 + \tfrac{1}{2}j)a_j \, . \qquad (63)$$

A polynomial of rank r is one whose terms are all of rank r.

There is a polynomial conservation law with nontrivial conserved density T_r for each positive integral rank r (and corresponding flux X_r of rank r+1).

Historically it is to be noted that Korteweg and de Vries [19] chose to

exhibit their equation in conservation law form i.e.

$$\frac{\partial\eta}{\partial t} = \frac{3}{2}\frac{\sqrt{g}}{\ell}\frac{\partial}{\partial x}(\tfrac{1}{2}\,\eta^2 + \tfrac{2}{3}\,\alpha\eta + \tfrac{1}{3}\,\sigma\,\frac{\partial^2\eta}{\partial x^2}) \, . \qquad (64)$$

η surface elevation above equilibrium level

α small arbitrarily constant related to the uniform motion of the liquid

g gravitational constant

σ $\ell^2/3 - T\ell/\rho g$

ρ liquid density

T surface capillary tension

The KdV equation describes the evolution of long water waves down a canal of rectangular cross-section. The KdV equation can be rewritten as

$$u_t - 6uu_x + u_{xxx} = 0 \qquad (65)$$

by the rescaling transformations

$$t' = \tfrac{1}{2}\sqrt{\frac{g}{\ell\sigma}}\,t \, , \quad X' = -\frac{x}{\sqrt{\sigma}} \, , \quad u = -\tfrac{1}{2}\eta - \tfrac{1}{3}\alpha \qquad (66)$$

where we have dropped the primes.

The KdV equation is Galilean invariant. (Muira [20]). Conservation laws can be used for deriving 'a priori' estimates and to obtain integrals of the motion. Three first conservation laws for the KdV equation are

$$u_t + (-3u^2 + u_{xx})_x = 0 \qquad (67a)$$

$$(u^2)_t + (-4u^3 + 2uu_{xx} - u_x^2)_x = 0 \qquad (67b)$$

$$(u^3 + \tfrac{1}{2}u_x^2)_t + (-\tfrac{9}{2}u^8 + 3u^2u_{xx} - 6uu_x^2 + u_xu_{xxx} - \tfrac{1}{2}u_{xx}^2)_x = 0 \qquad (67c)$$

The first is the KdV equation in conservation law form. The second is obtained by multiplying by 2u and the third by multiplying by $3u^2 - u_{xx}$ and algebraically manipulating the differentiations.

These conserved densities can be interpreted as mass, momentum and energy conservations for some physical systems. The infinity of conserved densities beyond these first three do not seem to have any physical interpretation. The proof of the existence of an infinite member of polynomial conservation laws was given by Kruskal, Muira, Gardner and Zabusky [18], Gardner [21] and Kruskal [22]. Muira [23] studied the polynomial conservation laws of a general class of KdV equations

$$W_t - 6W^pW_x' + W_{xxx} = 0 \quad p=1,2,\dots \tag{68}$$

This was also the starting point for the inverse scattering method for the exact solution of the KdV equations.

1.7 Conservation laws for generalized KdV equations

If we consider the class of equations

$$\frac{\partial u}{\partial t} + u^p \frac{\partial u}{\partial x} + \frac{\partial^r u}{\partial x^r} = 0 \tag{69}$$

(with all coefficients equal to 1, without loss of generality) Kruskal and Muira [22] obtained the following results:

a) if r is even (Burger's equation p=1, r=2, then there exists only one polynomial conservation law i.e. the equation itself)

b) If r is odd i.e. r=2q+1, q=0,1 there exist always three conservation laws for any p ≥ 0 with the conserved densities given by

$$\begin{cases} T_1 = u \\ T_2 = u \\ T_3 = u^{p+2} + \tfrac{1}{2}(-1)^q(p+1)(p+2)\left(\frac{\partial^q u}{\partial x^q}\right)^2 \end{cases} \tag{70}$$

For q=0, we define a new variable $v = u^p+1$ to get the equation

$$v_t + vv_x = 0 \tag{71}$$

which always has infinitely many conservation laws.

For the case p=0, the equation is linear and there are infinitely many conservation laws with conserved densities

$$T = (\partial^n u/\partial x^n)^2 \quad n=0,1,2,\dots \tag{72}$$

The cases p=1 and p=2 with q=1 correspond to the KdV equation or the modified KdV equation respectively. All the other cases have only the three (polynomial) conservation laws.

Table 1

q \ p	0	1	2	p ≥ 3
0	∞	∞	∞	∞
1	∞	∞	∞	3
q ≥ 2	∞	3	3	3

The results and concepts developed for the KdV equation like conservation laws have been extended to other nonlinear pde's like the nonlinear Schrödinger equation

$$i\psi_t + \psi_{xx} + |\psi|^2\psi = 0 \tag{73}$$

and the sine-Gordon equation (see Lamb [24])

$$u_{xt} = \sin u \tag{74}$$

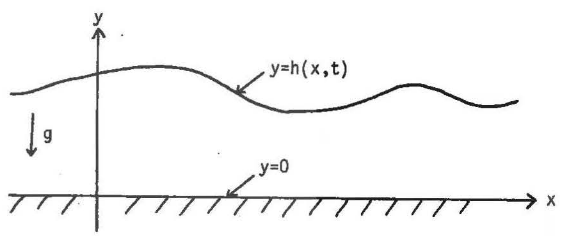
or

$$u_{tt} - u_{xx} + \sin u = 0 . \tag{75}$$

1.8 Benney's equations and their conservation laws

These are fully nonlinear long-wave equations describing the motion over a flat bottom of a two dimensionally inviscid fluid with a free surface in a gravitational field in the long-wave approximation.

The equations have the form



$$u_x + v_y = 0 \tag{76.1}$$

$$u_t + uu_x + vu_y + gh_x = 0 \tag{76.2}$$

$$h_t + u_o h_x - v_o = 0, \; y = h(x,t) \tag{76.3}$$

subject to boundary conditions

$$\left.\begin{array}{l} h_t + u_o h_x - v_o = 0 \\ \\ p = P_o = \text{constant} \end{array}\right\} \; y = h(x,t) \tag{76.4}$$

and $v = 0$ at $y = 0$.

The flat rigid bottom is denoted by $y = 0$ and the free surface by $y=h(x,t)$ (Benney [25]). Here u and v are the horizontal and vertical components of velocity, ρ is density.

If one defines the following integrals

$$A_n(x,t) = \int_0^{h(x,t)} u(x,y,t)^n dy, \; n=0,1,2,\ldots \quad (A_o(x,t) = h(x,t)) \tag{77}$$

it was shown by Benney [25] that this system of equations possesses an infinite number of conservation laws of the form

$$T_t + X_x = 0 \tag{78}$$

where T the conserved density and -X, the flux, are polynomials in the A_n n=0,1,2,... . Muira [23] proved that the fully nonlinear long-wave equations possess an infinite number of local conservation laws in the form

$$T_t + X_x + Y_y = 0 \tag{79}$$

where T the local conserved densities and -X,-Y the local fluxes are polynomials in n and the A_n.

Eqn (76.1) is already in conservation form. Multiplying (76.1) by u and adding to (76.2) yields a second conservation law

$$u_t + (u^2+gh)_x + (uv)_y = 0. \tag{80}$$

To find the third conservation law multiply (76.2) by u giving

$$(\tfrac{1}{2}u^2)_t + (\tfrac{1}{3}u^3)_x + (\tfrac{1}{2}u^2v)_y - \tfrac{1}{2}u^2v_y + gh_x u = 0. \tag{81}$$

Using (76.1) to replace v_y by $-u_x$ and integrating the last term by parts and again using (76.1) we obtain

$$(\tfrac{1}{2}u^2)_t + (\tfrac{1}{3}u^3+ghu)_x + [(\tfrac{1}{2}u^2+gh)v]_y = 0 \tag{82}$$

These three conservation laws have the usual interpretation of conserva= tion of mass, momentum and energy respectively.

Benney [25] obtained the following result for these equations: If $u(x,y,t)$ and $h(x,t)$ are continuous functions with continuous first-order partial derivatives for $-\infty < x < +\infty$, $0 \le y \le h$, and satisfy (76.1)-(76.4) then

$$(A_n)_t + (A_{n+1})_x + ug\, A_{n-1}h_x = 0 \quad n=0,1,2,\ldots \quad A_0=h. \tag{83}$$

For other extensions see Muira [26].

2. DISCRETE APPROXIMATIONS OF CONSERVATION LAWS

2.1 The Arakawa approach for an 'a priori' method for enforcing conservation laws on the discrete approximation of a simple fluid-dynamics equation

We shall introduce Arakawa's method [27] by considering the vorticity equation

$$\frac{\partial\zeta}{\partial t} + V.\nabla\zeta = 0, \quad \zeta = \nabla^2\psi \tag{84}$$

where the velocity V is assumed to be nondivergent i.e. it can be expressed as

$$V = k \times \nabla\psi \tag{85}$$

where ψ is a stream function and the vorticity is

$$\zeta = k\cdot\nabla\times V \tag{86}$$

i.e.

$$\zeta = \nabla^2\psi \tag{87}$$

and substituting (86) into (84). We obtain

$$\frac{\partial}{\partial t}\nabla^2\psi = \frac{\partial}{\partial t}\zeta = J(\nabla^2\psi,\psi) = J(\zeta,\psi) = -J(\psi,\nabla^2\psi). \tag{88}$$

This equation gives the local change in vorticity as a result of advection by a two-dimensional nondivergent velocity.

If the Gauss divergence theorem is applied to (84) over a closed region and there is no flux at the boundaries or the region is closed, the right-hand side of (84) vanishes. This means

$$\frac{\partial\bar{\zeta}}{\partial t} = 0 \tag{89}$$

where $\bar{\zeta}$ means the integral over the region, so that we obtain that the total (or mean) vorticity is conserved. If (84) is multiplied by ζ we obtain

$$\frac{\partial}{\partial t}\left(\frac{\zeta^2}{2}\right) = -\tfrac{1}{2}\nabla\cdot(\zeta^2 V). \tag{90}$$

Application of the Gauss divergence theorem again shows that the total or mean square vorticity called *enstrophy* is also conserved.

Finally, multiplying (88) by ψ, we obtain

$$\psi\partial(\nabla^2\psi)/\partial t = -\psi\nabla\cdot(\zeta V) \tag{91}$$

Modifying each term as follows

$$\psi\partial(\nabla\cdot\nabla\psi)/\partial t = \psi\nabla\cdot\nabla\frac{\partial\psi}{\partial t} = \nabla\cdot\left(\psi\nabla\frac{\partial\psi}{\partial t}\right) - \nabla\psi\cdot\nabla\frac{\partial\psi}{\partial t} \tag{92}$$

$$\psi\nabla\cdot(\zeta V) = \nabla\cdot(\psi\zeta V) - \zeta V\cdot\nabla\psi \tag{93}$$

and substituting these expressions in (91) gives

$$\nabla\psi\cdot\nabla\frac{\partial\psi}{\partial t} = \nabla\cdot\left(\psi\nabla\frac{\partial\psi}{\partial t}\right) + \nabla\cdot(\psi\zeta V) - \zeta V\cdot\nabla\psi\,. \tag{94}$$

The term $-\zeta V\cdot\nabla\psi$ vanishes identically since

$$V \perp \text{ to } \nabla\psi\,. \tag{95}$$

Also

$$\nabla\psi\cdot\nabla\frac{\partial\psi}{\partial t} = -\tfrac{1}{2}\,\partial\nabla\psi\cdot\nabla\psi/\partial t \tag{96}$$

where $\nabla\psi\cdot\nabla\psi$ is twice the kinetic energy. When the Gauss theorem is applie the right-hand side of (94) vanishes and we obtain

$$\partial(\overline{\nabla\psi\cdot\nabla\psi})/\partial t = 0 \qquad (97)$$

which means that the *mean kinetic energy* is conserved over a closed region. This means that for the vorticity equation (84) over a closed region, mean kinetic energy, mean vorticity and enstrophy are conserved.

In a famous paper published in 1959, using the vorticity equation, Phillips [28] showed that the leapfrog centred space-differencing scheme leads to nonlinear instability with a catastrophic 'blow-up' in kinetic energy in the short wavelenghts range. This is contrary to the results of Fjørtoft [29] who has shown that, in the continuous case, total kinetic energy is conserved and also that there is no cascade of energy towards short wavelengths in the continuous case, i.e. the average *wave number is also conserved.*

Phillips [28] found that a mere reduction in grid-size or time increment did not eliminate the nonlinear instability. The instability could how= ever be controlled by periodically removing wavelengths 4d and smaller. This was done either by the application of a space filter

$$\bar{A} = (1+k\nabla^2)^n A \qquad (98)$$

or by an artificial diffusing term such as $D\nabla^2 A$, where D is the eddy-diffusivity, or finally by using a selective finite-difference scheme.

All these methods are not very satisfying as they distort the solution by affecting longer waves in the process of a long-term numerical inte= gration.

Arakawa [27] sought to prevent a false cascade of energy and enstrophy by maintaining the conservation of mean enstrophy, kinetic energy and vorticity for the barotropic vorticity equation for a time-continuous space

difference discretization. He started with the barotropic vorticity equa tion written in Jacobian form

$$\frac{\partial\zeta}{\partial t} = -J(\psi,\zeta) \qquad (99)$$

by considering each of the products

$$\zeta J(\psi,\zeta) \text{ and } \psi J(\psi,\zeta) \qquad (100)$$

with several different finite difference Jacobians to determine whether they conserve kinetic energy or enstrophy.

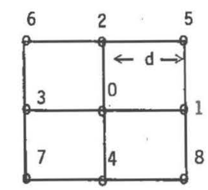
The essence of the Arakawa 'a priori' method will be exposed now.

The analytic Jacobian can be expressed as

$$J(\psi,\zeta) = \frac{\partial\psi}{\partial x}\frac{\partial\zeta}{\partial y} - \frac{\partial\psi}{\partial y}\frac{\partial\zeta}{\partial x} = \frac{\partial}{\partial x}\left(\psi\frac{\partial\zeta}{\partial y}\right) - \frac{\partial}{\partial y}\left(\psi\frac{\partial\zeta}{\partial x}\right) =$$

$$= \frac{\partial}{\partial y}\left(\zeta\frac{\partial\psi}{\partial x}\right) - \frac{\partial}{\partial x}\left(\zeta\frac{\partial\psi}{\partial y}\right) \qquad (101)$$

Using the following 9 point stencil



we can obtain the following finite difference Jacobians

$$J^{++} = [(\psi_1-\psi_3)(\zeta_2-\zeta_4) - (\psi_2-\psi_4)(\zeta_1-\zeta_3)]/4d^2 \qquad (102)$$

$$J^{+x} = [\psi_1(\zeta_5-\zeta_8)-\psi_3(\zeta_6-\zeta_7)-\psi_2(\zeta_5-\zeta_6)+\psi_4(\zeta_8-\zeta_7)]/4d^2 \qquad (103)$$

$$J^{x+} = [\zeta_2(\psi_5-\psi_6)-\zeta_4(\psi_8-\psi_7)-\zeta_1(\psi_5-\psi_8)+\zeta_3(\psi_6-\psi_7)]/4d^2 , \qquad (104)$$

The superscript + or x denotes the points from which the finite differenc approximations for the derivatives of ψ and ζ (in that order) are formed

A + symbol indicates use of the points (1,2,3,4) and the x symbol indicates the use of points (5,6,7,8). All the Jacobians (102-104) are of the general bilinear form

$$J_{ij}(\psi,\zeta) = \sum_{k,\ell,m,n}^{-1,0,+1} c_{i,j,k,\ell,m,n}\, \psi_{i+k,j+\ell}\, \zeta_{i+m,j+n}\,. \qquad (105)$$

The coefficients c can be determined so that the finite difference discretization possesses as many of the conservation properties of the analytic (continuous) differential equation as desired. If (99) is multiplied by ζ_{ij} the left-hand side represents the local rate of change of ζ^2_{ij}.

In this case the right-hand side may then be written as

$$\sum_{m,n}^{-1,0,1} a_{i,j,i+m,j+n}\, \zeta_{i+m,j+n}\, \zeta_{i,j}\,. \qquad (106)$$

The rate of change of ζ^2 at the point (i,j) may be considered to be the consequence of interaction between the given point and adjacent points. Also the change of ζ^2 at point (i+m,j+n) will receive a corresponding contribution from point ζ_{ij}. A net change of mean square vorticity (i.e. non-conservation) may be avoided if the gain at the point (i,j) due to interaction with point (i+m,j+n) is exactly balanced by a corresponding loss at (i+m,j+n) due to point (i,j). This occurs if the coefficients a fulfil the condition

$$a_{i,j,i+m,j+n} = -a_{i+m,j+n,i,j} \qquad (107)$$

Consider now the gain in square vorticity at point o due to the value at point 1 and vice versa. The results, excluding the term $4d^2$ are

$$\zeta_o\, J_o^{++} \sim -\,\zeta_o\, \zeta_1(\psi_2-\psi_4) + \text{other terms} \qquad (108)$$

$$\zeta_1\, J_1 \sim \zeta_1\, \zeta_0(\psi_5-\psi_8) + \text{other terms}\,. \qquad (109)$$

Since the terms do not cancel and no other opportunity for the product $\zeta_o\,\zeta_1$ occurs over the grid, the sum $\overline{\zeta J^{++}}$ cannot vanish in general and $\bar{\zeta}^2$ will not be conserved i.e.

$$\partial\overline{\zeta^2}/\partial t \neq 0\,.$$

Similarly

$$\zeta_o\, J_o^{+x} \sim \zeta_o\, \zeta_5(\psi_1-\psi_2) + \ldots \qquad (110)$$

$$\zeta_5\, J_5^{+x} \sim \zeta_5\, \zeta_0(\psi_2-\psi_1) + \ldots \qquad (111)$$

and

$$\zeta_o\, J_o^{x+} \sim -\,\zeta_o\, \zeta_1(\psi_5-\psi_8) + \ldots \qquad (112)$$

$$\zeta_1\, J_1^{x+} \sim \zeta_1\, \zeta_0(\psi_2-\psi_4) + \ldots\,. \qquad (113)$$

Arakawa [27] found that J^{+x} does conserve $\bar{\zeta}^2$ since the contributions at adjacent points exactly cancel one another, while a comparison of terms comprising ζJ^{++} and ζJ^{x+} shows that they are opposite in sign. So it is clear that

$$J^{+x} + \tfrac{1}{2}(J^{++} + J^{x+}) \text{ conserves } \overline{\zeta^2}. \qquad (114)$$

A similar examination of the finite difference approximation of $\psi J(\psi,\zeta)$ shows that

$$J^{x+} + \tfrac{1}{2}(J^{++} + J^{+x}) \text{ conserves } \bar{K}. \qquad (115)$$

By combining (114) and (115) it is seen that the Arakawa Jacobian

$$\tfrac{1}{3}\,(J^{++} + J^{+x} + J^{x+}) \qquad (116)$$

conserves enstrophy $\bar{\zeta}^2$, mean vorticity $\bar{\zeta}$ and mean kinetic energy $\bar{K}$. Arakawa [27] has shown that his Jacobian also conserves the *mean wave number*. As a result of these discrete conservation properties, the Arakaw

Jacobian prevents the continued growth of very short waves, which is charac= teristic of nonlinear instability. Aliasing is still present in the form of phase errors but these also result from linear finite-difference trun= cation-errors. Lilly [30] has given a rigorous proof by spectral methods that Arakawa's quadratic conserving scheme eliminates the type of nonlinear instability demonstrated by Phillips [28]. The conservative properties of the Arakawa Jacobian have been established for the time continuous case, but not for time discretization. For instance when the time derivative is approximated with the leapfrog time differencing scheme and the Arakawa Jacobian is used, it follows that

$$\overline{\zeta_{o,n}(\zeta_{o,n+1} - \zeta_{o,n-1})} = 2\Delta t\ \overline{\zeta_{o,n} J_{o,n}} = 0. \qquad (117)$$

Hence

$$\overline{\zeta_{o,n}\ \zeta_{o,n+1}} = \overline{\zeta_{o,n}\ \zeta_{o,n-1}}\ . \qquad (118)$$

But this is not the correct conservation property for mean square vorti= city, which should be written as

$$\overline{\zeta^2_{o,n+1}} = \overline{\zeta^2_{o,n}} = \overline{\zeta^2_{o,n-1}}\ ,\ \text{etc.} \qquad (119)$$

If the time differencing gives

$$\zeta_{o,n} = (\zeta_{o,n+1} + \zeta_{o,n-1})/2 \qquad (120)$$

the above (119) property holds but this requires a rather complicated implicit scheme. We shall come back to this shortcoming in another sec= tion. It has been demonstrated by Jespersen [31] that the Arakawa Jacobian is a special case of a finite-element method using rectangular 9-noded elements.

2.2 Generalization of Arakawa's scheme to the shallow-water equations (Grammeltvedt [32])

We write the equations for a barotropic, frictionless, divergent flow with a free surface (shallow-water equations) as

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv + \frac{\partial \varphi}{\partial x} = 0 \qquad (121a)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu + \frac{\partial \varphi}{\partial y} = 0 \qquad (121b)$$

$$\frac{\partial \varphi}{\partial t} + \frac{\partial(\varphi u)}{\partial x} + \frac{\partial(\varphi v)}{\partial y} = 0\ . \qquad (121c)$$

The velocities u and v are in the x and y directions respectively and φ=gh is the geopotential of the fluid where g is the acceleration of gravity and h is the height of the free surface; f is the Coriolis para= meter.

These equations conserve the integrals of mass, total energy and enstroph in the channel $0 \le x \le D$, $0 \le y \le L$, $t \ge 0$. In the following we will use a regular grid with horizontal spacing $\Delta x = \Delta y = \Delta$, and time increment Δt.

To derive the discretized finite-difference equations we will use the Shuman sum and difference operators

$$\alpha_x = \frac{1}{\Delta}\,[\alpha(x_i + \tfrac{\Delta}{2}) - \alpha(x_i - \tfrac{\Delta}{2})] \qquad (122)$$

$$\bar{\alpha}^x = \tfrac{1}{2}\,[\alpha(x_i + \tfrac{\Delta}{2}) + \alpha(x_i - \tfrac{\Delta}{2})] \qquad (123)$$

$$\bar{\alpha}^{2x} = \tfrac{1}{2}[\alpha(x_i + \Delta) + \alpha(x_i - \Delta)] \qquad (124)$$

$$J^{xx} = \overline{(\bar{\alpha}^x)}^x = \alpha + \frac{\Delta^2}{4}\alpha_{xx} = \tfrac{1}{4}[\alpha(x_i+\Delta)+\alpha(x_i-\Delta)+2\alpha(x_i)]\ . \qquad (125)$$

A second and fourth-order finite-difference approximation to the first anc second derivatives of α can then be written as

$$\frac{\partial\alpha}{\partial x} = \bar{\alpha}^{x}_{x} + O(\Delta^2) \tag{126}$$

$$\frac{\partial\alpha}{\partial x} = \tfrac{4}{3}\ \bar{\alpha}^{x}_{x} - \tfrac{1}{3}\ \bar{\alpha}^{2x}_{2x} + O(\Delta^4) \tag{127}$$

$$\frac{\partial^2\alpha}{\partial x^2} = \alpha_{xx} + O(\Delta^2) \tag{128}$$

$$\frac{\partial^2\alpha}{\partial x^2} = \tfrac{4}{3}\ \alpha_{xx} - \tfrac{1}{3}\ \alpha_{2x2x} + O(\Delta^4)\ . \tag{129}$$

Now the Arakawa scheme for the two-dimensional vorticity equation describing frictionless, non-divergent flow in a closed domain is written in the Shuman notation

$$\bar{\zeta}^{t}_{t} + \tfrac{1}{3}\ [\bar{\psi}^{x}_{x}\ \bar{\zeta}^{y}_{y} - \bar{\psi}^{y}_{y}\ \bar{\zeta}^{x}_{x} + \overline{(\psi\bar{\zeta}^{y}_{y})_{x}}^{x} - \overline{(\psi\bar{\zeta}^{x}_{x})_{y}}^{y} + \tag{130}$$

$$+ \overline{(\bar{\psi}^{x}_{x}\zeta)_{y}}^{y} - \overline{(\bar{\psi}^{y}_{y}\zeta)_{x}}^{x} = 0\ .$$

If we now use the vorticity components u and v in the x and y direction respectively, instead of the stream function ψ and the vorticity ζ, by using the following relationships

$$u = -\frac{\partial\psi}{\partial y} \text{ and } v = \frac{\partial\psi}{\partial x} \tag{131}$$

i.e.

$$u = -\psi_y \text{ and } v = \psi_x \tag{132}$$

and if we use second-order finite-difference approximations and $\zeta = v_x - u_y$ for the vorticity, Grammeltvedt [32] after some lengthy algebra derived the following generalization of Arakawa's scheme for the barotropic equa= tions (shallow-water equations).

$$\bar{u}^{t}_{t} + \tfrac{1}{3}\ \{(\bar{u}^{x}\ \bar{u}^{x})_{x} + \overline{\tfrac{1}{2}(u(u+\Delta^2\nabla^2\bar{u}^{y}_{y}))_{x}}^{x} +$$

$$+ 2\bar{v}^{xy}\ \bar{u}^{y}_{y} + \overline{\bar{v}^{x}\ u_{y}}^{y}\} - f\bar{v}^{xy} + \varphi_{x} = 0 \tag{133a}$$

$$\bar{v}^{t}_{t} + \tfrac{1}{3}\ \{2\bar{u}^{xy}\ \bar{v}^{x}_{x} + \overline{\bar{u}^{y}\ v_{x}}^{x} + (\bar{v}^{y}\ \bar{v}^{y})_{y} + \overline{\tfrac{1}{2}(v(v+\Delta^2\nabla^2 v))_{y}}^{y} + f\bar{u}^{xy} + \varphi_{y} = 0 \tag{133b}$$

and

$$\bar{\varphi}^{t}_{t} + (\bar{\varphi}^{x}\ u)_{x} + (\bar{\varphi}^{y}v)_{y} = 0 \tag{133c}$$

where

$$\nabla^2 u = u_{xx} + u_{yy} \tag{134}$$

and the locations of u,v and φ in the grid are given by

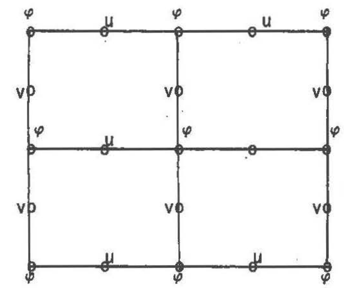


Fig.2

For the non-divergent flow, the generalized Arakawa scheme will conserve mean kinetic energy, mean vorticity and enstrophy.

2.3 Two general conservation laws for shallow-water equations

If we consider the equations describing the motion in a homogeneous in= compressible fluid with a free surface having bottom topography, we obtain

$$\frac{\partial V}{\partial t} + q\underline{k} \times V^* + \nabla\cdot(K+\Phi) = 0 \tag{135}$$

$$\frac{\partial h}{\partial t} + \nabla\cdot V^* = 0\ . \tag{136}$$

Here q is the potential vorticity and V^* the mass flux defined by

$$q = (f+\zeta)/h \qquad (137)$$

$$V^* = hV \qquad (138)$$

where V is the horizontal velocity, t the time, f the Coriolis parameter, ζ the vorticity, $\zeta = k\cdot\nabla X V$, $\underline{k}$ is the vertical unit vector, ∇ the horizontal del operator, h the vertical extent of a fluid column above the bottom surface, K the kinetic energy per unit mass, $\frac{1}{2}V^2$, g the gravitation acceleration and h_s the bottom surface height and

$$\Phi = g(h+h_s) . \qquad (139)$$

If we multiply (135) by V^* and combine the results with (136), we obtain the equation for the time change of total kinetic energy

$$\frac{\partial}{\partial t}(hK) + \nabla\cdot(V^*K) + V^*\cdot\nabla\Phi = 0 . \qquad (140)$$

If we multiply (136) by Φ we obtain the equation for the time change of potential energy,

$$\frac{\partial}{\partial t}(\tfrac{1}{2}gh^2+ghh_s) + \nabla\cdot(V^*\Phi) - V^*\cdot\nabla\Phi = 0 . \qquad (141)$$

Summation of (140) and (141) yields the equation of the conservation of total energy

$$\frac{\partial}{\partial t}\,\overline{[h(K + \tfrac{1}{2}gh + gh_s)]} = 0 \qquad (142)$$

where the overbar denotes the mean over a finite domain with no inflow or outflow through the boundaries. The vorticity equation for this fluid motion is obtained from (135) and can be written as

$$\frac{\partial}{\partial t}(hq) + \nabla\cdot(V^*q) = 0 . \qquad (143)$$

Subtracting (136) times q from (143) and dividing by h gives

$$\frac{\partial q}{\partial t} + V\cdot\nabla q = 0 . \qquad (144)$$

Now hq times (144) plus $\frac{1}{2}q^2$ times (136) gives the equation for the time change of potential enstrophy (Ertel's theorem [33]).

$$\frac{\partial}{\partial t}(\tfrac{1}{2}hq^2) + \nabla\cdot(V^*\tfrac{1}{2}q^2) = 0 \qquad (145)$$

which leads to the conservation-law of potential enstrophy

$$\frac{\partial}{\partial t}\overline{(\tfrac{1}{2}hq^2)} = 0 . \qquad (146)$$

Arakawa and Lamb [34] derived, after a very lengthy algebra, a potential enstrophy and energy conserving scheme for the shallow-water (S-W) equations with bottom topography for general flow. They have shown that their scheme suppresses the spurious energy cascade towards short-wave numbers and that the overall flow regime is adequately represented. This potential enstrophy 'a priori' conserving scheme was generalized by Arakawa and Lamb to a spherical grid while K.Takano [34] of UCLA derived a fourth-order accurate version of the scheme, both for square and spherical grids.

2.4 The global barotropic (S-W) equations and their conservation laws (spherical coordinates)

These equations can be written as:

$$\frac{\partial u}{\partial t} - \frac{z}{\cos\theta}(\varphi v\cos\theta) + \frac{1}{a\cos\theta}\frac{\partial}{\partial\lambda}(\varphi + \tfrac{1}{2}(u^2+v^2)) = 0 \qquad (147a)$$

$$\frac{\partial v}{\partial t} + z\varphi u + \frac{1}{a}\frac{\partial}{\partial\theta}(\varphi + \tfrac{1}{2}(u^2+v^2)) = 0 \qquad (147b)$$

$$\frac{\partial\varphi}{\partial t} + \frac{1}{a\cos\theta}\{\frac{\partial}{\partial\lambda}(\varphi u) + \frac{\partial}{\partial\theta}(\varphi v\cos\theta)\} = 0 \qquad (147c)$$

where

$$z = \{f + \frac{1}{a\cos\theta}(\frac{\partial v}{\partial\lambda} - \frac{\partial}{\partial\theta}(u\cos\theta))\}/\varphi \qquad (148)$$

is the potential vorticity.

a - radius of the globe

θ - longitude

λ - latitude

This system has the following conservation laws

$$\frac{\partial}{\partial t}\int_0^{\pi}\int_0^{2\pi} \varphi \cos\theta \, a^2 \, d\lambda d\theta = 0 \tag{149}$$

conservation of 'mass'

$$\frac{\partial}{\partial t}\int_0^{\pi}\int_0^{2\pi} \frac{1}{a\cos\theta}\left\{\frac{\partial v}{\partial \lambda} - \frac{\partial}{\partial\theta}(u\cos\theta)\right\}\cos\theta \, a^2 d\lambda d\theta = 0 \tag{150}$$

conservation of vorticity

$$\frac{\partial}{\partial t}\int_0^{\pi}\int_0^{2\pi}\left\{\frac{\varphi^2}{2} + \varphi \tfrac{1}{2}(u^2+v^2)\right\}a^2 \cos\theta \, d\lambda d\theta = 0 \tag{151}$$

conservation of energy

$$\frac{\partial}{\partial t}\int_0^{\pi}\int_0^{2\pi} t^2\varphi \, a^2\cos\theta \, d\lambda d\theta = 0 \tag{152}$$

conservation of potential enstrophy.

Sadourny [35] has shown that formal conservation of potential enstrophy is more important than formal conservation of total energy, in that potential enstrophy conserving models of the S-W equations are inherently more stable and maintain more realistic energy spectra.

In this case we choose to conserve mass, vorticity and potential enstrophy. Away from the poles such a finite element discretization is (Burridge [36])

$$\frac{\partial u}{\partial t} = \bar{z}^{\theta}\, \overline{V\cos\theta}^{\theta\lambda} - \frac{1}{a\cos\theta}\delta_\lambda(\varphi+e) \tag{153}$$

$$\frac{\partial v}{\partial t} = \bar{\bar{Z}}^{\lambda}\, \bar{\bar{V}}^{\theta\lambda} - \frac{1}{a}\delta_\theta(\varphi+e) \tag{154}$$

$$\frac{\partial \varphi}{\partial t} = -\frac{1}{a\cos\theta}(\delta_\lambda U + \delta_\theta(V\cos\theta)) \tag{155}$$

where the zonal and meridianal mass fluxes are given by

$$U = \bar{\varphi}^{\lambda}u, \; V = \bar{\varphi}^{\theta}v \text{ respectively} \tag{156}$$

and

$$e = \tfrac{1}{2}(\overline{u^2}^{\lambda} + \frac{1}{\cos\theta}\overline{(v^2\cos\theta)}^{\theta} \text{ (kinetic energy)} \tag{157}$$

$$z = \frac{\lambda}{\overline{\varphi\cos\theta}^{\theta\lambda}}\{\overline{f\cos\theta}^{\theta} + \delta_\lambda v - \delta_\theta(u\cos\theta)\} \tag{158}$$

and δ_λ, δ_θ are the two-point centred derivatives acting on immediate neighbours along θ or λ directions. These finite differences used with the regular staggered grid illustrated in fig 2 conserves total mass, vorticity and the potential enstrophy, apart from boundary fluxes at or near the poles.

Although energy is not conserved formally by these finite difference forms it turns out that the model conserves energy very accurately. Near the poles a special form of finite differences is required to enforce the same conservation laws. Details can be found in Burridge [36].

2.5 Conservation laws for baroclinic primitive equations models

The equations for a dry adiabatic version of a baroclinic model are in the normalized vertical pressure coordinate σ

$$\frac{\partial p_s}{\partial t} + \nabla_\sigma\cdot(p_s V) + \frac{\partial}{\partial\sigma}(p_s\dot{\sigma}) = 0 \quad \text{(conservation of mass)}$$

$$\frac{\partial V}{\partial t} + zk \times V + \dot{\sigma}\frac{\partial V}{\partial\sigma} + \nabla_\sigma(\varphi+e) + RT\nabla_\sigma \ell n p_s = 0 \quad \text{(momentum)} \tag{160}$$

$$\frac{\partial}{\partial t}(p_s c_p T) + \nabla_\sigma \cdot (p_s V c_p T) + \frac{\partial}{\partial \sigma}(p_s \dot{\sigma} c_p T) - \frac{RTw}{\sigma} = 0 \quad \text{(thermodynamics)} \qquad (161)$$

$$\frac{\partial \varphi}{\partial \ell n \sigma} = -RT \quad \text{(hydrostatic equation)} \qquad (162)$$

where the vertical coordinate is σ and is defined by

$$\sigma = p/p_s \quad \dot{\sigma} = d\sigma/dt \qquad (163)$$

where p_s is the surface pressure.

At the surface $p=p_s$ hence $\sigma=1$ and $\dot{\sigma}=0$; at the top of the atmosphere $p=0$ hence $\sigma=0$ and $\dot{\sigma}=0$.

$$z = \frac{1}{p_s} f + \frac{1}{a \cos \theta}\left[\frac{\partial v}{\partial \lambda} - \frac{\partial}{\partial \theta}(a \cos \theta)\right] \qquad (164)$$

$$e = \tfrac{1}{2}(u^2+v^2) = \tfrac{1}{2}V.V \qquad (165)$$

$$w = \dot{p} = p_s \dot{\sigma} + \sigma\left\{\frac{\partial p_s}{\partial t} + V.\nabla_\sigma p_s\right\}. \qquad (166)$$

Here

T - is the temperature

R - gas constant for air

c_p- specific heat at constant pressure

The boundary conditions are

$$(p_s \dot{\sigma})_{\sigma=0} = 0 \qquad (167)$$

$$(p_s \dot{\sigma})_{\sigma=1} = 0 \qquad (168)$$

('no flux' conditions).

For this baroclinic model we require mass and total energy conservation.

Integration of the continuity equation gives

$$\frac{\partial p_s}{\partial t} = -\int_0^1 \nabla_\sigma \cdot (p_s V) d\sigma - [p_s \dot{\sigma}]_0^1 = -\nabla_\sigma \cdot \int_0^1 p_s V d\sigma . \qquad (169)$$

The divergence term in (169) integrates to zero over the globe which means that the total mass is conserved.

A 'kinetic' energy equation can be derived from the momentum equations in the form

$$p_s \frac{\partial e}{\partial t} + p_s V \cdot \nabla_\sigma e + p_s \dot{\sigma} \frac{\partial e}{\partial \sigma} + p_s V \cdot (\nabla_\sigma \varphi + RT\nabla_\sigma \ell n p_s) = 0 . \qquad (170)$$

Using the continuity equation (159), (170) can be written as

$$\frac{\partial}{\partial t}(p_s e) + \boxed{\nabla_\sigma \cdot (p_s V e) + \frac{\partial}{\partial \sigma}(p_s \dot{\sigma} e)} + p_s V \cdot (\nabla_\sigma \varphi + RT\nabla_\sigma \ell n p_s) = 0 . \qquad (171)$$

In order to construct the total energy equation the pressure gradient terms must be expressed in another form. Using the continuity equation (159) we have

$$p_s V \cdot \nabla_\sigma \varphi = \nabla_\sigma \cdot (p_s V \varphi) + \frac{\partial}{\partial \sigma}\left(\varphi\left(\sigma \frac{\partial p_s}{\partial t} + p_s \dot{\sigma}\right)\right) - \frac{\partial \varphi}{\partial \sigma}\left(\sigma \frac{\partial p_s}{\partial t} + p_s \dot{\sigma}\right). \qquad (172)$$

Combining (171) and (172) we have

$$\frac{\partial}{\partial t}(p_s e) + \boxed{\nabla_\sigma \cdot (p_s V e) + \frac{\partial}{\partial \sigma}(p_s \dot{\sigma} e) + \nabla_\sigma (p_s V \sigma) + \frac{\partial}{\partial \sigma}\left(\left(\sigma \frac{\partial p_s}{\partial t} + p_s \dot{\sigma}\right)\varphi\right)} - \frac{\partial \varphi}{\partial \sigma}\left\{\sigma \frac{\partial p_s}{\partial t} + p_s \dot{\sigma}\right\} + RTp_s V \cdot \nabla_\sigma \ell n p_s . \qquad (173)$$

Now

$$-\frac{\partial \varphi}{\partial \sigma}\left(\sigma \frac{\partial p_s}{\partial t} + p_s \dot{\sigma}\right) + RTp_s V \cdot \nabla_\sigma \ell n p_s =$$

$$= \frac{RT}{\sigma}\left(\sigma \frac{\partial p_s}{\partial t} + p_s \dot{\sigma}\right) + RTV \cdot \nabla_\sigma p_s = \frac{RTw}{\sigma} . \qquad (174)$$

Addition of the kinetic energy equation and the flux form of the thermo= dynamic equation and integration from $\sigma=0$ to 1 gives

$$\frac{\partial}{\partial t}\{p_s\varphi_s + \int_0^1 p_s(e+c_pT)d\sigma\} + \nabla_\sigma\cdot\int_0^1 p_sV(e+c_pT+\varphi)d\sigma = 0. \qquad (175)$$

This equation expresses the energy conservation for the σ system (159)-(163). The divergence term integrates to zero over the whole globe - i.e. we have that the total energy, potential plus kinetic, is conserved.

2.6 The discretized version of the baroclinic primitive equations which maintains conservation of energy and mass (Burridge [36])

The vertical grid (see Fig.3) is such that the primary variables V and T are carried at integral levels, while the vertical velocity $\dot{\sigma}$ and the geopotential φ are carried at half integer levels. The variable vertical grid spacing $\Delta\sigma_k$ is defined by

$$\Delta\sigma_k = \sigma_{k+\frac{1}{2}} - \sigma_{k-\frac{1}{2}}\ . \qquad (176)$$

Vertical grid Fig.3

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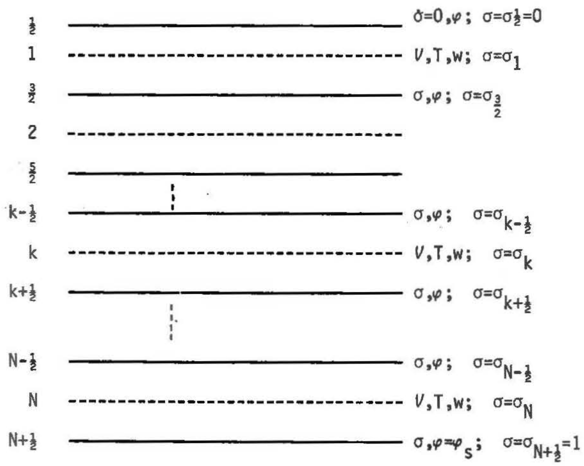

2.6.1 The equation of continuity

The vertically discretized continuity equation takes the form

$$\frac{\partial p_s}{\partial t} + \nabla_\sigma\cdot(p_sV_k) + \frac{\Delta(p_s\dot{\sigma})_k}{\Delta\sigma_k} = 0\ . \qquad (177)$$

Multiplying (177) by $\Delta\sigma_k$ and summing (instead of integration) from k=1,2,...,k gives

$$\sigma_{k+\frac{1}{2}}\frac{\partial p_s}{\partial t} + \sum_{k=1}^{k}\nabla_\sigma\cdot(p_sV_k)\Delta\sigma_k + p_s\dot{\sigma}_{k+\frac{1}{2}} = 0 \qquad (178)$$

$$(\text{as } p_s\dot{\sigma}_{\frac{1}{2}} = 0)\ .$$

For k=N we have

$$\frac{\partial p_s}{\partial t} = -\sum_{k=1}^{N}\nabla_\sigma\cdot(p_sV_k)\Delta\sigma_k = -\nabla_\sigma\cdot\sum_{k=1}^{N}\nabla_\sigma\cdot(p_sV_k)\Delta\sigma_k\ . \qquad (179)$$

This equation is the finite-difference analogue of (169) which obviously conserves the total mass of the model atmosphere.

2.6.2 Energy conservation

The vertically discretized finite-difference momentum equation is

$$\frac{\partial V_k}{\partial t} + zk\times V_k + \tfrac{1}{2}\{\frac{\dot{\sigma}_{k+\frac{1}{2}}(V_{k+\frac{1}{2}}-V_k)+\dot{\sigma}_{k-\frac{1}{2}}(V_k-V_{k-1})}{\Delta_\sigma\sigma_k}\}$$

$$+ \nabla_\sigma\{\varphi_k+e_k\} + RT_k\nabla_\sigma \ell n p_s = 0 \qquad (180)$$

where

$$\varphi_k = \alpha_k\varphi_{k+\frac{1}{2}} + \beta_k\varphi_{k-\frac{1}{2}} \text{ with } \alpha_k+\beta_k = 1 \qquad (181)$$

α_k,β_k are weights.

The finite difference kinetic energy equation is

$$p_s \frac{\partial e_k}{\partial t} + \frac{1}{2}\,\frac{(p_s\dot{\sigma}_{k+\frac12} V_{k+\frac12}\cdot V_k - p_s\dot{\sigma}_{k-\frac12} V_k\cdot V_{k-1})}{\Delta\sigma_k}$$

$$- \frac12 V_k\cdot V_k \frac{\Delta(p_s\dot{\sigma})_k}{\Delta\sigma_k} + p_s V_k\cdot\nabla_\sigma e_k + p_s V_k\cdot\{\nabla_\sigma\varphi_k + RT_k\nabla_\sigma \ell n p_s\} = 0. \quad (182)$$

This equation may be rewritten in flux form by using the finite difference continuity equation

$$\frac{\partial}{\partial t}(p_s e_k) + \nabla_\sigma\cdot(p_s V_k e_k) + \Delta_\sigma\Big(\frac{p_s\dot{\sigma}\tilde{V^2}^{\sigma}}{2}\Big)_k$$

$$+ p_s V_k\cdot\{\nabla_\sigma\varphi_k + RT_k\nabla_\sigma \ell n p_s\} = 0 \quad (183)$$

where

$$(\tilde{V^2}^{\sigma})_{k+\frac12} = V_{k+\frac12}\cdot V_k \text{ (geometric mean)}. \quad (184)$$

For the hydrostatic equation we write

$$\Big(\frac{\Delta_\sigma\varphi}{\Delta\, \ell n\sigma}\Big)_k = -RT_k \quad (185)$$

and we obtain a finite-difference analogue of (172)

$$p_s V_k \nabla_\sigma\varphi_k = \nabla_\sigma\cdot\{p_s V_k\varphi_k\} - \varphi_k\nabla_\sigma\cdot(p_s V_k)$$

$$= \nabla_\sigma\cdot\{p_s V_k\varphi_k\} + \varphi_k\Big\{\frac{\partial p_s}{\partial t} + \frac{\Delta_\sigma(p_s\dot{\sigma})_k}{\Delta_{\sigma k}}\Big\}$$

$$= \nabla_\sigma\cdot\{p_s V_k\varphi_k\} + \frac{\Delta_\sigma(\varphi(\sigma\frac{\partial p_s}{\partial t} + p_s\dot{\sigma}))}{\Delta\sigma_k} \quad (186)$$

$$\frac{-(\Delta_\sigma\varphi)_k}{\Delta\sigma_k}\Big[\beta_k\Big(\sigma\frac{\partial p_s}{\partial t} + p_s\dot{\sigma}\Big)_{k+\frac12} + \alpha_k\Big(\sigma\frac{\partial p_s}{\partial t} + p_s\dot{\sigma}\Big)_{k-\frac12}\Big]$$

$$\alpha_k + \beta_k = 1$$

as

$$\varphi_k(A_{k+\frac12} - A_{k-\frac12}) = \varphi_{k+\frac12}A_{k+\frac12} - \varphi_{k-\frac12}A_{k-\frac12} -$$

$$-(\varphi_{k+\frac12} - \varphi_{k-\frac12})(\beta_k A_{k+\frac12} + \alpha_k A_{k-\frac12}). \quad (187)$$

If we require total energy conservation then the form of the last term in (186) is a constraint in our choice of the w term

$$\frac{Tw}{\sigma} = \frac{1}{cp}\frac{RTw}{\sigma} \quad (188)$$

in the thermodynamic equation whose discretized form is

$$p_s\frac{\partial T_k}{\partial t} + p_s V_k\cdot\nabla_\sigma T_k + p_s\,\frac12\Big\{\frac{\dot{\sigma}_{k+\frac12}(T_{k+1}-T_k)+\dot{\sigma}_{k-\frac12}(T_k-T_{k-1})}{\Delta\sigma_k}\Big\}$$

$$-\frac{1}{cp}\Big[\frac{RTw}{\sigma}\Big]_k = 0. \quad (189)$$

If the w-term is chosen so as to maintain energy conservation we have

$$\frac{1}{cp}\Big[\frac{RTw}{\sigma}\Big]_k = \frac{1}{cp}\Big[-\frac{(\Delta\varphi)_k}{\Delta\sigma_k}\Big\{\beta_k\Big(\sigma\frac{\partial p_s}{\partial t} + p_s\dot{\sigma}\Big)_{k+\frac12}$$

$$+ \alpha_k\Big(\sigma\frac{\partial p_s}{\partial t} + p_s\dot{\sigma}\Big)_{k-\frac12}\Big\} + RT_k\, V_k\cdot\nabla p_s\Big]$$

$$= \frac{1}{cp}\Big[RT_k\Big(\frac{\Delta\,\ell n\sigma}{\Delta\sigma}\Big)_k\Big\{\beta_k\Big(\sigma\frac{\partial p_s}{\partial t} + p_s\dot{\sigma}\Big)_{k+\frac12}$$

$$+ \alpha_k\Big(\sigma\frac{\partial p_s}{\partial t} + p_s\dot{\sigma}\Big)_{k-\frac12}\Big\} + RT_k\, p_s\, V_k\cdot\nabla\ell n p_s\Big]. \quad (190)$$

The term $[\Delta\ell n\sigma/\Delta\sigma]_k$ can be interpreted as a definition of $(1/\sigma_k)$ for the baroclinic model.

Our remaining degree of freedom is in the choice of the weights α_k and β_k subject to $\alpha_k+\beta_k=1$. The ECMWF operational model uses $\alpha_k=\beta_k=\frac12$.

More about this topic can be found in an article by Arakawa and Lamb [37] describing the UCLA general circulation model.

Mesinger [38] developed a series of enstrophy and energy conserving finite-difference schemes for the horizontal advection. An explicit potential vorticity conservation by finite-difference schemes framed in generalized vertical coordinates was developed by Bleck [39].

3. 'A POSTERIORI' METHODS FOR ENFORCING CONSERVATION LAWS IN DISCRETIZED FLUID-DYNAMICS EQUATIONS

All the 'a priori' numerical schemes modelling the discrete conservation laws result in rather complicated finite-difference schemes that are difficult to generalize to fluid-dynamics problems of interest.

A different approach is to enforce the conservation relationships expli= citly by modifying the forecast field values, at each time step of the numerical integration. Sasaki ([40], [41], [42]) proposed such a variational approach and applied it to conserve total energy and mass in one and two-dimensional shallow-water equations models on a rotating plane.

Bayliss and Isaacson [43], Isaacson and Turkel [44] and Isaacson [45] presented a simple method of making any finite difference scheme conser= vative with respect to any quantity. In their approach the conservative constraints were linearized about the predicted values by means of a gradient method for modifying the predicted values at each time step of the numerical integration.

Isaacson et al [46] and Isaacson et al [47] have implemented the same technique in terms of simultaneous conservation for the shallow-water equations over a sphere, taking into account orography effects.

Their approach has been tested by Kalnay-Rivas et al [48] with enstrophy as the conserved quantity. Kalnay-Rivas [48], [49] found that the use of an enstrophy conserving scheme can be successfully replaced by using a fourth-order quadratically conserving scheme on a global domain, com= bined with the periodic application of a 16th-order Shapiro filter re= moving waves shorter than four times the grid size before they attain finite amplitude.

Navon [50] has shown that it is possible to achieve stable long-term integrations of the shallow-water equations using the above mentioned techniques. In these lecture notes a modified Sasaki variational approach to enforce conservation of total mass and potential enstrophy will be developed and then a modified Bayliss-Isaacson technique also designed to enforce conservation of potential enstrophy and total mass, will be described.

A new approach based on viewing the problem of enforcing conservation laws in discretized models as a nonlinearly constrained optimization problem solved by an augmented Lagrangian penalty method will be men= tioned (Navon-de Villiers [51]).

3.1 <u>Formulation of the modified Sasaki variational method for enforcing conservation of potential enstrophy and mass</u>

3.2 <u>The numerical variational algorithm</u>

3.3 <u>The Bayliss-Isaacson algorithm</u>

[*Editor's note*: As this material has already been published elsewhere ([50], §2-§4), copyright restrictions do not allow its repetition. It should nevertheless be regarded as an integral part of the present survey, and for that reason the paragraph headings have been retained.]

4. INTEGRAL CONSTRAINTS FOR THE TRUNCATED SPECTRAL EQUATION

We shall illustrate in brief the properties of the spectral method by using the barotropic nondivergent equation

$$\frac{\partial \zeta}{\partial t} = -V\cdot\nabla(\zeta+f) \qquad (191)$$

with the usual meaning for all symbols.

For a non-divergent flow on the spherical globe we obtain

$$\zeta = \underline{k} \cdot \nabla \times V = \nabla^2 \psi \tag{192}$$

where ψ is a scalar stream function. We write

$$V = (U,V) \tag{193}$$

where

$$U = \frac{\cos\theta}{a}\frac{\partial\psi}{\partial\varphi} \text{ and } V = \frac{1}{a}\frac{\partial\psi}{\partial\lambda} \tag{194}$$

Equation (191) is written in spherical polar coordinates as

$$\frac{\partial}{\partial t}(\nabla^2\psi) = \frac{1}{a^2 \cos\varphi}\left[\frac{\partial\nabla^2\psi}{\partial\lambda}\frac{\partial\psi}{\partial\varphi} - \frac{\partial\psi}{\partial\lambda}\frac{\partial\nabla^2\psi}{\partial\varphi}\right] - \frac{2\Omega}{a^2}\frac{\partial\psi}{\partial\lambda} \tag{195}$$

where φ, λ, a and Ω denote respectively, latitude, longitude and the earth's radius and rotation rate.

Using an expansion of ψ in orthogonal surface spherical harmonics, a truncated expression for the stream function is

$$\psi(\lambda,\varphi) = a \sum_{m=-J}^{+J} \sum_{\ell=|m|}^{|m|+J} \psi_\ell^m(t)\, Y_\ell^m(\lambda,\varphi) \tag{196}$$

$$Y_\ell^m(\lambda,\varphi) = p_\ell^m(\sin\varphi)e^{im\lambda} \tag{197}$$

where $p_\ell^m(\sin\varphi)$ is the associated Legendre polynomial of the first kind normalized to unity, and the truncation parameter J denotes a rhomboidal wave number truncation, while m denotes a planetary wave number and ℓ-m denotes a meridional wave number.

The nonlinear advection term is written as

$$C = \frac{1}{a^2\cos\varphi}\left(\frac{\partial\nabla^2\psi}{\partial\lambda}\frac{\partial\psi}{\partial\varphi} - \frac{\partial\nabla^2\psi}{\partial\varphi}\frac{\partial\psi}{\partial\lambda}\right) = \sum_{m=-J}^{+J}\sum_{\ell=|m|}^{|m|+J} C_\ell^m\, Y_\ell^m \tag{198}$$

and

$$C_\ell^m = \frac{i}{2}\sum_{m_1}\sum_{\ell_1}\sum_{m_2}\sum_{\ell_2}\{\ell_2(\ell_2+1)-\ell_1(\ell_1+1)\}\psi_{\ell_1}^{m_1}\psi_{\ell_2}^{m_2}L(\ell_1,\ell_2,\ell;m_1,m_2,m). \tag{199}$$

Here

$$L(\ell_1,\ell_2,\ell;m_1,m_2,m) = \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} p_\ell^m \left\{m_1 p_{\ell_1}^{m_1}\frac{dp_{\ell_2}^{m_2}}{d\varphi} - m_2 p_{\ell_2}^{m_2}\frac{dp_{\ell_1}^{m_1}}{d\varphi}\right\} d\varphi. \tag{200}$$

Lorenz [58] proved that as a consequence of the non-aliased truncation, the integral invariants of the non-divergent barotropic vorticity equation (191) are also valid for the solution to the truncated spectral equa= tions. Lorenz [58] proved this for plane geometry using a representa= tion in terms of double Fourier series, while Platzman [59] did the proof for spherical geometry. Merilees [60] explicitly illustrated the result that the truncated set of spectral equations retains as invariant the domain integral of kinetic energy via conservative redistribution of energy within each individual interaction. For an extensive review on this issue see Machenhauer [61]).

While spectral approximations conserve the gross characteristics of the energy spectrum and a systematic energy cascade towards higher wavenumbers is not possible, a certain *blocking* of energy in the highest wavenumbers occurs. In actual numerical integrations the integral constraints are of course only approximately valid due to time truncation and round-off errors.

For three dimensional models the property of the spectral approximation of non-aliased truncation of the non-linear terms implies quasi-conser= vation of energy in adiabatic, friction-free numerical integrations. (Merilees [60]). Weigle [62] has made a detailed study of the conserva= tion properties of a spectral shallow-water equations model while for more general sigma-levels models Bourke [63] and Hoskins and Simmons [64] found that energy is very nearly conserved during adiabatic, friction-free integration.

The blocking phenomenon resulting from neglecting interactions involving components outside the truncation limits was observed by Gordon and Stern [65] and Bourke [63].

In long-term low resolution spectrally truncated models the amplitudes of small-scale components will grow unrealistically large and a scale-selective damping parametrization is required.

5. CONSERVATION LAWS AND FINITE-ELEMENT DISCRETIZATION

Here we will only refer to the Galerkin finite-element method. This method (see Cullen [66]) has a property of satisfying certain conserva= tion laws.

If we set

$$w = \frac{\partial u}{\partial x} \qquad (201)$$

and express $w = \Sigma w_n X_n(x,y)$ where X_n are basis functions, the Galerkin method gives

$$\int_\Omega (w - \frac{\partial u}{\partial x}) X_n = 0 \qquad (202)$$

using the basic functions as test functions.

If we multiply the n-th member of the system (202) by u_n and sum over n we get

$$\int (w - \frac{\partial u}{\partial x}) \Sigma\, u_n\, X_n = 0 \qquad (203)$$

so that

$$\int (w - \frac{\partial u}{\partial x}) u = 0 \qquad (204)$$

exactly.

In many problems, this equation expresses conservation of energy or mo= mentum.

In general, any conservation law which can be expressed by multiplying the differential equation by *one* variable and integrating will be satis= fied exactly in the Galerkin method. Raymond and Garder [67] however find that conservation is a disadvantage on irregular grids.

The two-stage highly accurate Galerkin method does not conserve 'energy' (Cullen [68]). It is likely that this is because the aliasing effect is removed in this finite element integration and that instability is less likely. Lee et al [69] considered the Boussinesq inviscid problem by finite-element discretization and found out that a method which con= served the quadratic quantities was most stable in terms of time integra= tion, whereas the standard advective form, which does not conserve any of the three quantities conserved by the Boussinesq equation namely total energy ($\tilde{E}$), total temperature ($\tilde{T}$) and total temperature squared ($\tilde{T}^2$), performed very poorly in time integrations.

5.1 Conservation properties of the Boussinesq equation for finite-elemen discretization

The equations for an inviscid Boussinesq fluid in a two dimensional regio Ω, with boundary Γ are:

$$\rho\, [\frac{\partial \underline{u}}{\partial t} + \underline{u} \cdot \nabla \underline{u}] = -\nabla p - \rho \underline{g} \gamma T \text{ in } \Omega \qquad (205a)$$

$$\nabla \cdot \underline{u} = 0 \text{ in } \Omega \qquad (205b)$$

$$\frac{\partial T}{\partial t} + \underline{u} \cdot \nabla T = 0 \text{ in } \Omega \qquad (205c)$$

where ρ is density, $\underline{u}$ the velocity, p the pressure, g the acceleration due to gravity, γ the volumetric coefficient of thermal expansion and T the temperature.

For the case of a contained flow the boundary conditions are

$$\underline{u}\cdot\underline{n} = 0 \text{ on } \Gamma \qquad (206)$$

where $\underline{n}$ is the outward pointing unit normal vector on Γ.

From (205c) we have

$$\frac{\partial}{\partial t} T^n + \underline{u}\cdot\nabla T^n = 0 \qquad (207)$$

so that

$$\frac{d}{dt}\int_\Omega T^n = \int_\Omega \frac{\partial T^n}{\partial t} = -\int_\Omega \underline{u}\cdot\nabla T^n = -\int_\Omega \nabla\cdot(\underline{u}T^n) = -\int_\Gamma (\underline{u}\cdot\underline{n})T^n = 0. \qquad (208)$$

Thus any integral of T is conserved.

From (205a) we find

$$\rho(\frac{\partial}{\partial t}\tfrac{1}{2}\underline{u}^2 + \underline{u}\cdot\nabla\,\tfrac{1}{2}\underline{u}^2) = -\underline{u}\cdot\nabla p - \rho\underline{u}\cdot\underline{g}\gamma T \qquad (209)$$

so

$$\frac{d}{dt}\int_\Omega \tfrac{1}{2}\,\rho\underline{u}^2 = -\int_\Omega \nabla\cdot(\tfrac{1}{2}\rho\underline{u}^2+p)\underline{u} - \int_\Omega \rho\underline{u}\cdot\underline{g}\gamma T =$$

$$= -\int_\Gamma (\underline{u}\cdot\underline{n})(\tfrac{1}{2}\rho\underline{u}^2+p) - \int_\Omega \rho\underline{u}\cdot\underline{g}\gamma T.$$

Now

$$\frac{d}{dt}\int_\Omega \rho\underline{r}\cdot\underline{g}\gamma T = \int_\Omega \rho\underline{r}\cdot\underline{g}\gamma\,\frac{\partial T}{\partial t} = -\int_\Omega \rho\underline{r}\cdot\underline{g}\gamma(\underline{u}\cdot\nabla T)$$

$$= -\rho\gamma\underline{g}\cdot\{\int_\Gamma(\underline{u}\cdot\underline{n})\underline{r}T - \int_\Omega [\underline{r}(\nabla\cdot\underline{u})+\underline{u}]T\} = \int \rho\gamma\underline{g}\cdot\underline{u}T \qquad (210)$$

where $\underline{r}$ is the position vector.

From (206),(208),(209) and (210) we get:

$$\frac{d}{dt}\int_\Omega (\tfrac{1}{2}\rho\underline{u}^2 + \rho\gamma\underline{g}\cdot\underline{r}T) = 0. \qquad (211)$$

This equation expresses the conservation-law of the total energy of the fluid.

In a breakthrough paper Cliffe [70] found a particular Galerkin discreti= zation (using two parameters α and β and different finite-element inter=

polation spaces for pressure and temperature) such that the total energy $\bar{E}$, the total temperature $\bar{T}$ and the total temperature squared ($\bar{T}^2$) were conserved. He used an eight-noded quadrilateral serendipity element for the pressure field, while for the temperature field a standard four-noded element was used. The weights were $\alpha=\beta=\frac{1}{2}$.

5.2 <u>The effect of time-discretization and other errors on conservation laws</u> (Cliffe [70], Sanz-Serna [71]).

We will first examine the effect of time-discretization. A semi-discrete equation can be written as

$$\dot{y} = f(y). \qquad (212)$$

To preserve the discrete conservation law we require a non-dissipative method of integration for (212). The simplest is the trapezoidal rule (Crank-Nicolson) leading to

$$y^{(n+1)} - y^{(n)} = \frac{\Delta t}{2}\,(f(y^{(n+1)}) + f(y^{(n)})). \qquad (213)$$

Lee et al [69]as well as Sánz-Serna [71] have observed that whilst (213) will preserve the conservation properties of linear quantities, for quadratic quantities which are conserved in the semi-discrete equation (213) the conservation properties are lost.

A second-order non-dissipative method, the midpoint rule, does however preserve the conservation properties of quadratic quantities i.e.

$$y^{(n+1)} - y^{(n)} = \Delta t\, f(\tfrac{1}{2}(y^{(n+1)} + y^{(n)})) \qquad (214)$$

Other sources of error are the solution of nonlinear systems of equa= tions. In exact arithmetic, linear conservation properties are preserved at each iteration while for quadratic quantities, conservation properties are only preserved in the limit of convergence.

If the region is not regular then isoparametric transformations of the boundary finite-elements lead to integrals which have to be evaluated by Gaussian integration schemes introducing *quadrature errors*. Finally, all conservation properties are affected by rounding-errors, which are usually quite small.

6. REFERENCES

[1] Eiseman, P.R., Stone, A.P., 1980. Conservation-laws of fluid-dynamics, a survey, *SIAM Review* Vol.22, No.1, pp.12-27.

[2] Wahlquist, H.D., Estabrook, F.B., 1975. Prolongation structures of nonlinear evolution equations I. *J.Math.Phys.* 16, No.1, pp.17.

[3] Manin, Yu, 1979. Algebraic aspects of nonlinear differential equations. *J.Soviet Math.* Vol.11, pp.1-122.

[4] Gary, J., 1979. Nonlinear instability in Numerical Methods Used in Atmospheric Models. Vol.II. *GARP publications series* No 17, WNO pp 476-499.

[5] Kalnay de Rivas, E., Hoitsma, D.,1979. The effect of accuracy, conservation and filtering on numerical weather forecasting. Fourth Conference on Numerical Weather Prediction. Silver-Spring Vol.AMS. pp 302-312.

[6] Mobbs, S.D. 1981. Some vorticity theorems and conservation-laws for non-barotropic fluids. *J.Fluid Mech.* Vol.108, pp 475-483.

[7] Lax,P.D., 1957. Hyperbolic Systems of conservation-laws II. *Comm. Pure Appl.Math.*, 10, pp 537-566.

[8] Liu, T.P.,1977. Linear and nonlinear large-time behaviour of solutions of general systems of hyperbolic conservation-laws. *Comm.Pure Appl.Math.* 30, pp. 767-796.

[9] Liu, T.P., 1979. Quasilinear hyperbolic systems. *Comm.Maths.Phys.* 68, pp. 141-172.

[10] Glium, J. 1965. Solutions in the large for nonlinear hyperbolic Systems of equations. *Comm.Pure Appl.Math.* 18, 697-715.

[11] Glium, J., Lax, P.D. 1970. Decay of solutions of nonlinear hyper= bolic conservation laws. AMS *Memoir* No.101.

[12] Anderson, J., Preiser, S., Rubin, E. 1968. Conservation-law form of the equations of hydrodynamics in curvilinear coordinate systems. *J.Comput.Phys.* Vol.2, pp 279-287.

[13] Flanders, H., 1963. *Differential forms*. Academic Press London and New York.

[14] Noether, E. 1918. Invariante Variationsprobléme Nacht Akad. Wiss. Göttingen. *Math.Phys Kl.* pp 238-257.

[15] Osborn, H. 1954. The existence of conservation-laws. Tech.Report No.27, Applied Mathematics and Statistics Laboratory, Stanford University, Stanford, California.

[16] Viviand, H. 1977. Formes conservatives des equations de la dynamiq des gaz. La Recherche Aerospatide, No.1 pp 65-68.

[17] Vinokut, M. 1974. Conservation equations of Gasdynamics in curvi= linear coordinate systems. *J.Comput. Phys.* Vol.14 pp 105-125.

[18] Kruskal, M.D., Muira, R.M., Gardner, C.S., Zabusky, N.J.1970. Korteweg-de Vries and Generalizations V. Uniqueness and poly= nomial conservation-laws. *J.Math.Phys.* Vol.11, No.3, pp. 952-960.

[19] Korteweg, D.J., de Vries, G. 1895. On the change of forms of long-waves advancing in a rectangular canal, and on a new type of long stationary waves. *Philos. Mag.* 39 pp 422-443.

[20] Muira, R.M. 1976. The Korteweg-de Vries equation. A survey of results. *SIAM Review*, Vol.18, No.3, pp 412-457.

[21] Gardner, C.S. 1971. Korteweg-de Vries equation and generalizations. IV. The Korteweg-de Vries equation as a Hamiltonian system. *J.Math.Phys.* 12, pp.1548-1551.

[22] Kruskal, M.D., Muira,R.M. Korteweg-de Vries equation and generali= zations VII. Nonexistence of polynomial conservation-laws.

[23] Muira,R.M. 1968. Korteweg-de Vries equation and generalizations. I. A remarkable explicit nonlinear transformation. *J. Math. Phys.* Vol.9. pp 1202-1204.

[24] Lamb, J.G.L. Jr. 1974. Börklund transformations of certain nonli= near evolution equations. *J.Math.Physics*. Vol.15, p.2157-2165.

[25] Benney, D.J. Long-waves in Nonlinear Wave Motion. A.C.Newell ed. American Math.Soc. Providence pp 49-60.

[26] Muira, R.M. 1974. Conservation-laws for the fully nonlinear long-wave equations. *Studies Appl. Math.* Vol.53, No.1, pp.45-56.

[27] Arakawa, A. 1966. Computational design for long-term integrations of the equations of atmospheric motion. *J.Comput.Phys.* 1, 119-143.

[28] Phillips, N.A.1959. An example of nonlinear computational insta= bility. The Atmosphere and the Sea in Motion. Rossbly Memo= rial Volume, Rockefeller Institute Press, 501-504 (509pp).

[29] Fjørtoft, R.1953. On the changes in the spectral distribution of kinetic energy for two dimensional, nondivergent flow. *Tellus J.* pp 225-230.

[30] Lilly, D.K. 1965. On the computational stability of numerical solutions of time-dependent nonlinear geophysical fluid dynamics problems. *Mon.Wea.Rev.* 93, pp.11-26.

[31] Jespersen, D.C. 1979. Arakawa's method is a finite-element method. *J. Comput. Phys.* 16, pp.383-390.

[32] Grammeltvedt, A. 1969. A survey of finite-difference schemes for the primitive equations for a barotropic fluid. *Mon.Wea.Rev.* Vol.97, No.5, pp 384-404.

[33] Ertel,H. 1942. Einen Neur Hydrodynamischer Wirbelsatz. *Met.Zeit.* 59. pp 277-281.

[34] Arakawa, A., Lamb,V.R. 1981. A potential enstrophy and energy conserving scheme for the shallow-water equations. *Mon.Wea Rev.* Vol.109, pp.18-36.

[35] Sadourny, R. 1975. The dynamics of finite-difference models of the shallow-water equations. *J.Atmos.Sci.* 32, pp.680-689.

[36] Burridge, D.M.1979. Some aspects of large scale numerical modelling of the atmosphere in ECMWF Seminar, Vol.2, Dynamic Meteorlogy and Numerical Weather Prediction. pp.1-78.

[37] Arakawa, A.,Lamb, V.R.1977. Computational design of the basic dynamical processes of the UCLA General Circulation Model: In *Methods of Computational Physics*, Vol.17, J.Chang Ed. pp 174-265.

[38] Mesinger, F.1981. Horizontal advection schemes of a staggered grid An enstrophy and energy conservation model. *Mon.Wea.Rev.*Vol.10[illegible] pp 467-478.

[39] Bleck,R.1979. Difference equations in generalized vertical coor= dinates. Part II. Potential Vorticity Conservation. *Contr. Atmos.Phys*. Vol.52, No.2, pp.95-105.

[40] Sasaki, Y. 1975. Variational design of finite-difference schemes for the initial-value problem of conservative system. Dept Meteorlogy, Univ. Oklahoma, 31 pp.

[41] Sasaki, Y.1976. Variational design of finite-difference schemes for initial value problems with an integral invariant. *J. Comput.Phys*. 21, pp.270-278.

[42] Sasaki, Y.,1977. Variational design of finite-difference schemes for initial-value problems with a global divergent barotropic model. *Contrib.Atmos.Phys*. Vol.50, pp.184-189.

[43] Bayliss, A., Isaacson,E. 1975. How to make your algorithm conser= vative. *Not Amer.Math.Soc*. August 1975, A594-A595.

[44] Isaacson,E,Turkel,E.,1976. Conservation of integral invariants for numerical solutions. Abstract of talk given at SIAM meeting, October 1976, Atlanta,Go.

[45] Isaacson,E.,1977. Integration Schemes for Long Term Calculation in Advances in Computer Methods for Partial Differential Equations. A Vichvetsky Ed. AICA pp 251-255.

[46] Isaacson,E.,Harten,A.,Marchesin,D.,Zwas,G.,1979b. Implicit versus explicit fourth-order schemes for the shallow-water equations over a sphere. Presented at SIAM meeting Nov.1979,Denver, Co.

[47] Isaacson, E.,Marchesin,D.,Zwas,G.,1979a. Numerical methods for meteorlogy and climatology. Fourth NASA Weather and Climate Program Science Review. F E Kreiss Ed. NASA Conf.Pub.1076. Paper 31,pp 183-190.

[48] Kalnay Rivas,E., Bayliss,A.,Storch,J.,1977. The 4-th order GISS model of the global atmosphere. *Contrib.Atmos.Phys*. 50, pp.306-311.

[49] Kalnay Rivas,E.,1976. Numerical experiments with fourth order con= servative finite differences. *Annalen der Meteorologie* 11 (Extended abstract).

[50] Navon, I.M.,1981. Implementation of 'a posteriori' methods for enforcing conservation of potential enstrophy and mass in discretized shallow-water equations models. *Mon.Wea.Rev*. Vol. 109. No.5.,pp 946-958.

[51] Navon,I.M., de Villiers, R.1981. An augmented Lagrangian penalty method for enforcing conservation of integral invariants via constrained optimization (To appear).

[52] Sasaki, Y.,1969. Proposed inclusion of time-variation terms, ob= servational and theoretical in numerical variational objective analysis. *J.Met.Soc.Japan*. Vol.47,No.2,pp 115-127.

[53] Sasaki, Y, 1970a. Some basic formalisms in numerical variational analysis. *Mon.Wea.Rev*. 98, 875-883.

[54] Sasaki, Y. 1970b. Numerical variational analysis formulated under the constraints as determined by long-wave equations as a low-pass filter. *Mon.Wea.Rev*. 98, pp 884-898.

[55] Sadourny, R.,1973. Quasi-inviscid dynamics of the truncated shallow-water equations.Proc. Symp.Diff.and Spectral Methods for Atmosphere and Ocean Dynamics Problems USSR. Acad.Sci. Siberian Branch, Computing Centre, Novosibirska, Part II, pp.61-87.

[56] Sadourny,R.,1980. Conservation-laws,quasi-two dimensional turbulence and numerical modelling of large-scale flows. Seminar ECMWF 1979. Vol.2 Dynamic Meteorology and Numerical Weather Prediction.pp 16

[57] Fairweather, G.,Navon,I.M.,1980. A linear ADI method for the shallow-water equations. *J.Comp.Phys.* 37.pp.1-18.

[58] Lorenz, E.N.,1960. Maximum simplification of the dynamic equations *Tellus* 12, pp 243-254.

[59] Platzman,G.W.,1960. The spectral form of the vorticity equation. *J.Meteorology,* 17, pp 635-644.

[60] Merilees, P.E.,1968. The equations of motion in spectral form. *J.Atmos.Sci.* 25, pp 736-743.

[61] Machenhauer, B.,1979. The Spectral Method in Numerical Methods used in Atmosphere Models.Vol.II. *GARP Publication Series* Vol.17,WMO, pp 124-275.

[62] Weigle, W.F.,1972. Energy conservation and the truncated spectral form of the primitive equations for a one-layer model. NCAR Cooperative thesis No.26. University of Michigan and NCAR.

[63] Bourke, W.,1974. A multi-level spectral model I. Formulation and hemispherical integrations. *Mon.Wea.Rev.*,102, pp 687-701.

[64] Hoskins, J.,Simmons, A.J.,Simmons,B.J.,1974. The development of spectral models in the UK. University Atmospheric Modelling Group. GARP WGNE Report No.7, pp 94-99.

[65] Gordon,T.,Stern,W.,1974. Spectral modelling at GFDL. GARP WGNE Report No.7, pp 46-82.

[66] Cullen, M.J.P.,1979. The finite element method in Numerical Methods used in Atmospheric Models. Vol.II. *GARP publications Series* Vol.17, WMO, pp 302-337.

[67] Raymond, W.H.,Garder, A.,1976. Selective damping in a Galerkin method for solving wave problems with a variable grid. *Mon.Wea Rev.* 104, pp 1583-1590.

[68] Cullen, M.J.P.,1979a. A finite-element method for a nonlinear initial value problem. *J.Inst.Math.Applications,* 13, pp 233-277.

[69] Lee, R.L.,Gresho,P.M.,Chan,S.T.,Sani,R.L.,1980. A comparison of several conservation forms for finite-element formulations of the imcompressible Navier-Stokes or Boussinesq equations. Proc.Third Int.Conf.on Finite Elements in Flow Problems, Banff,Canada, 1980.

[70] Cliffe, K.A.,1981. On conservative finite-element formulations of the inviscid Boussinesq equations. *Int.Jour Methods in Fluids.* Vol.1, pp 117-127.

[71] Sanz-Serna,J.M.,1981. An explicit finite-difference scheme with exact conservation properties. NRIMS Technical Report TWISK 234, CSIR, Pretoria, 21 pp.